# Indefinite Integration

## INDEFINITE INTEGRATION Section - 1

### 1.1 Definition

The indefinite integral of a function $f(x)$ with respect to $x$ is another function $g(x)$ whose derivative is $f(x)$.

i.e $g'(x) = f(x)$ $\Rightarrow$ indefinite integral of $f(x)$ is $g(x)$.

In Mathematical Notation, we write :

$$\int f(x)dx = g(x) + C \text{ if and only if } g'(x) \qquad \left[as \frac{d}{dx}[g(x)+C] = g'(x)\right]$$

where $C$ is an arbitrary constant i.e., independent of $x$.

Here $\int$ is the integral sign, $f(x)$ is the intergrand, $x$ is the variable of integration, $dx$ is the element of integra -tion, $g(x)$ is the integral of the function $f(x)$ and the process is known as *Indefinite Integration*.

The indefinite integral of a function is also called as antiderivative of that function.

### 1.2 Properties of Integral Calculus - 1

(i) $\int k\, f(x)dx = k\int f(x)\,dx$

(ii) $\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots\ldots\ldots \pm f_n(x)]dx$

$$= \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \int f_3(x)dx \pm \ldots\ldots \pm \int f_n(x)dx$$

(iii) $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$

(iv) $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$

(v) $\int f(x)\,dx = g(x)$, then $\int f(ax+b)\,dx = \frac{1}{a}g(ax+b) + C$

### 1.3 Standard Results in Integration

As integration is the process of finding the anti-derivatives of various functions, we can derive integrals of various basic functions by using the definition of their derivatives.

For Example :

$$\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x\,dx = \sin x + C$$

$$\frac{d}{dx}(e^x) = e^x \quad \Rightarrow \quad \int e^x\, dx = e^x + C$$

$$\frac{d}{dx}\tan x = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x\, dx = \tan x + C$$

In this way we can derive the integrals of many basic functions and learn them as formulas which can readily be used to find the integrals of various complicated functions.

So from our experience in differentiation, we can derive the following results and can learn them as standard formulas.

### 1.3.1 Integrals of Basic Algebraic Functions

Use the expressions for derivatives of trigonometric functions, we have the following results :

(i) $\int \frac{1}{x^2}\, dx = -\frac{1}{x} + C$

(ii) $\int \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x} + C$

(iii) $\int \sqrt{x}\, dx = \frac{2}{3} x\sqrt{x} + C$

(iii) $\int \frac{1}{x^p}\, dx = \frac{-1}{(p-1)\, x^{p-1}} + C \quad (p \neq 1)$

In each of these results, verify that the derivatives of R.H.S is equal to the function that we are integrating. All these can also be derived as special cases of the following general results.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq 1)$$

For $n = -1$, we have : $\int \frac{1}{x} dx = \log|x| + C$

### 1.3.2 Integrals of Basic Trigonometric Functions

Using the expressions for derivatives of trigonometric functions, we have the following results :

(i) $\int \sin x\, dx = -\cos x + C$

(ii) $\int \cos x\, dx = \sin x + C$

(iii) $\int \sec^2 x\, dx = \tan x + C$

(iv) $\int \text{cosec}^2 x\, dx = -\cot x + C$

(v) $\int \sec x \tan x\, dx = \sec x + C$

(vi) $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$

### 1.3.3 Integrals of Basic Inverse Trigonometric and Exponential Functions

Similarly, recall the derivatives of inverse and exponential function to prove the following results :

(i) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C\,;\, |x| < 1$

(ii) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

(iii) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C\,;\, |x| > 1$

(iv) $\int a^x dx = \frac{a^x}{\log a} + C\,;\, a \neq 1,\, a > 0$

(v) $\int e^x \, dx = e^x + C$

## 1.4 Further Generalisation of Some Results using property 1.2 (v)

$$\int f(x)\, dx = g(x) + C \quad \Rightarrow \quad \int f(ax+b)\, dx = \frac{1}{a} g(ax+b) + C$$ where $a$ and $b$ are constants.

Using the above property, we can extend our list of basic formulas through the following results :

(i) $\int (ax+b)^n \, dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C$

(ii) $\int \frac{dx}{(ax+b)^2} = \frac{1}{a}\left(\frac{-1}{ax+b}\right) + C$

(iii) $\int \frac{dx}{\sqrt{ax+b}} = \frac{1}{a} 2\sqrt{ax+b} + C$

(iv) $\int \frac{dx}{(ax+b)^p} = \frac{1}{a} \frac{-1}{(p-1)(ax+b)^{p-1}} + C \; (p \neq 1)$

(v) $\int \frac{1}{ax+b} dx = \frac{1}{a} \log|ax+b| + C$

(vi) $\int \sin(ax+b)\, dx = -\frac{1}{a} \cos(ax+b) + C$

(vii) $\int \cos(ax+b)\, dx = \frac{1}{a} \sin(ax+b) + C$

(viii) $\int \sec^2(ax+b)\, dx = \frac{1}{a} \tan(ax+b) + C$

(ix) $\int \operatorname{cosec}^2(ax+b)\, dx = -\frac{1}{a} \cot(ax+b) + C$

(x) $\int \sec(ax+b) \tan(ax+b)\, dx = \frac{1}{a} \sec(ax+b) + C$

(xi) $\int \operatorname{cosec}(ax+b) \cot(ax+b)\, dx = -\frac{1}{a} \operatorname{cosec}(ax+b) + C$

(xii) $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$

## 1.5 Integrals of tan *x*, cot *x*, sec *x*, cosec *x*

All these integrals can be evaluated using the result :

$$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$$ [Also given in Section 1.2 (iii)]

i.e. $$\int \frac{\text{derivative of denominator}}{\text{denominator}} dx = \log|\text{denominator}| + C$$

(i) $\int \tan x \, dx = \int \frac{\sec x \tan x}{\sec x} dx$

$\Rightarrow \quad \int \tan x \, dx = \log|\sec x| + C$

(ii) $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \log|\sin x| + C$

(iii) $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{\sec^2 x \ \sec x + \tan x}{\sec x + \tan x} dx$

$\Rightarrow \quad \int \sec x\, dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(iv) $\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx = \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec} x - \cot x} dx$

$\Rightarrow \quad \int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C = \log|\tan x/2| + C$

### *Illustrating the Concepts :*

*Evaluate the following Integrals :*

(i) $\int sec^2(2-3x)\, dx$ (ii) $\int \frac{sin(2-3x)}{cos^2(2-3x)} dx$ (iii) $\int e^{2x-3} dx$

(iv) $\int sec(2-3x)\, dx$ (v) $\int \frac{1}{\sqrt{4x+1}} dx$ (vi) $\int \frac{dx}{(1-2x)^3}$

Hint : *Express Integrals in terms of standard results.*

(i) $\int \sec^2(2-3x)\, dx = \frac{-1}{3} \tan(2-3x) + C$

(ii) $\int \frac{\sin(2-3x)}{\cos^2(2-3x)} dx = \int \sec(2-3x)\tan(2-3x)\, dx$

$= \frac{1}{-3} \sec(2-3x) + C$

(iii) $\int e^{2x-3} dx = \frac{1}{2} e^{2x-3} + C$

(iv) $\int \sec(2-3x)\, dx$

$= \frac{1}{-3} \log|\sec(2-3x) + \tan(2-3x)| + C$

(v) $\int \frac{1}{\sqrt{4x+1}} dx = \frac{1}{4}\left(2\sqrt{4x+1}\right) + C$

(vi) $\int \frac{dx}{(1-2x)^3} = \frac{1}{-2}\left(\frac{-1}{2(1-2x)^2}\right) + C$

***Illustrating the Concepts :***

*Evaluate the following Integrals :*

(i) $\int \frac{x-1}{x+1}dx$ (ii) $\int \frac{x^2-1}{x^2+1}dx$ (iii) $\int \frac{x}{(2x+1)}dx$ (iv) $\int \frac{x^4+1}{x^2+1}dx$

(v) $\int \frac{x^7}{x+1}dx$ (vi) $\int \frac{x^3}{(x+1)^2}dx$ (vii) $\int \frac{ax+b}{cx+d}dx$

Hint: *Express numerator in terms of denominator.*

(i) $\int \frac{x-1}{x+1}dx = \int \frac{x+1-2}{x+1}dx = \int \left(1-\frac{2}{x+1}\right)dx = \int dx - 2\int \frac{dx}{x+1} = x - 2\log|x+1| + C$

(ii) $\int \frac{x^2-1}{x^2+1}dx = \int \frac{x^2+1}{x^2+1}dx - \int \frac{2dx}{x^2+1} = x - 2\tan^{-1}x + C$

(iii) $\int \frac{x}{(2x+1)^2}dx = \frac{1}{2}\int \frac{2x+1-1}{(2x+1)^2}dx = \frac{1}{2}\int \frac{dx}{(2x+1)} - \frac{1}{2}\int \frac{dx}{(2x+1)^2}$

$$= \frac{1}{2}\left(\frac{1}{2}\log|2x+1|\right) - \frac{1}{2}\left[\frac{1}{2}\left(\frac{-1}{2x+1}\right)\right] + C$$

(iv) $\int \frac{x^4+1}{x^2+1}dx = \int \frac{x^4-1}{x^2+1}dx + \int \frac{2}{x^2+1}dx = \int (x^2-1)\,dx + 2\int \frac{dx}{x^2+1} = x^3/3 - x + 2\tan^{-1}x + C$

(v) $\int \frac{x^7}{x+1}dx = \int \frac{x^7+1}{x+1}dx - \int \frac{dx}{x+1} = \int \frac{(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)}{x+1}dx - \log|x+1|$

$$= \frac{x^7}{7} - \frac{x^6}{6} + \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$$

(vi) $\int \frac{x^3}{(x+1)^2}dx = \int \frac{x^3+1}{(x+1)^2}dx - \int \frac{dx}{(x+1)^2} = \int \frac{x^2-x+1}{(x+1)}dx - \int \frac{dx}{(x+1)^2}$

$$= \int \frac{x^2+x}{x+1}dx + \int \frac{1-2x}{x+1}dx - \int \frac{dx}{(x+1)^2}$$

$$= \int xdx - 2\int \frac{x+1}{x+1}dx + \int \frac{3dx}{x+1} - \int \frac{dx}{(x+1)^2}$$

$$= \frac{x^2}{2} - 2x + 3\log|x+1| + \frac{1}{x+1} + C$$

(vii) $$\int \frac{ax+b}{cx+d}\,dx = \frac{a}{c}\int \frac{(cx+d)-\left(d-\frac{bc}{a}\right)}{cx+d}\,dx = \frac{a}{c}\int dx - \frac{a}{c}\int \frac{\left(d-\frac{bc}{a}\right)}{cx+d}dx$$

$$= \frac{ax}{c} - \left(\frac{ad-bc}{c^2}\right)\log|cx+d| + C$$

## Illustrating the Concepts :

*Evaluate the following Integrals :*

(i) $\int \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}\,dx$  (ii) $\int \sec^4 x \tan x\,dx$  (iii) $\int \frac{\log^n x}{x}\,dx$

(iv) $\int \frac{x}{(x^2+1)}\,dx$  (v) $\int \sin^5 x \cos x\,dx$

Hint : *Use* $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)^{n+1}]}{n+1} + C$

(i) $$\int \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}\,dx = \frac{1}{4}(\sin^{-1}x)^4 + C$$

(ii) $$\int \sec^4 x \tan x\,dx = \int \sec^3 x(\sec x \tan x)\,dx$$

$$= \frac{\sec^4 x}{4} + C$$

(iii) $$\int \frac{\log^n x}{x}\,dx = \frac{\log^{n+1}x}{n+1} + C$$

(iv) $$\int \frac{x}{(x^2+1)}\,dx = \frac{1}{2}\int \frac{1}{(x^2+1)^3}2x\,dx$$

$$= \frac{1}{2}\frac{(x^2+1)^{-2}}{-2} + C$$

(v) $$\int \sin^5 x \cos x\,dx = \frac{\sin^6 x}{6} + C$$

### *Illustrating the Concepts :*

*Evaluate the following Integrals :*

(i) $\int \frac{x^3}{1+x^4}dx$ (ii) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$ (iii) $\int \frac{e^x+1}{e^x-1}dx$

(iv) $\int \frac{x^3}{(x^2+1)^2}dx$ (v) $\int \frac{dx}{a+be^x}dx$ (vi) $\int \frac{\tan x+1}{\tan x-1}dx$

(vii) $\int \frac{\sec x}{\log(\sec x+\tan x)}dx$ (viii) $\int \frac{x^2-1}{x(x^2+1)}dx$ (ix) $\int \frac{dx}{x+\sqrt{x}}$

(x) $\int \frac{x}{(x^4+1)\tan^{-1}x^2}dx$ (xi) $\int \frac{dx}{x\log x \log\log x}$ (xii) $\int \frac{\sin 2x}{1+\sin^2 x}dx$

(xiii) $\int \frac{e^{x-1}+x^{e-1}}{e^x+x^e}dx$

Hint : *Use* $\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+C$.

(i) $\int \frac{x^3}{1+x^4}dx = \frac{1}{4}\int \frac{4x^3}{1+x^4}dx \;= \frac{1}{4}\log|1+x^4|+C$

(ii) $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx = \log|e^x+e^{-x}|+C$

(iii) $\int \frac{e^x+1}{e^x-1}dx \int \frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} = 2\int \frac{\frac{1}{2}\left[e^{x/2}+e^{-x/2}\right]}{e^{x/2}-e^{-x/2}}dx = 2\log|e^{x/2}-e^{-x/2}|+C$

(iv) $\int \frac{x^3}{(x^2+1)^2}dx = \int \frac{x^3+x}{(x^2+1)^2}dx - \int \frac{xdx}{(x^2+1)^2} = \int \frac{x}{x^2+1}dx - \int \frac{x}{(x^2+1)^2}dx$

$$= \frac{1}{2}\int \frac{2x}{x^2+1}dx - \frac{1}{2}\int \frac{2x}{(x^2+1)^2}dx = \frac{1}{2}\log|x^2+1| + \frac{1}{2}\left(\frac{1}{x^2+1}\right)+C.$$

(v) $\int \frac{dx}{a+be^x} = \int \frac{e^{-x}}{ae^{-x}+b}dx = -\frac{1}{a}\int \frac{-ae^{-x}}{ae^{-x}+b}dx \;= -\frac{1}{a}\log|ae^{-x}+b|+C$

(vi) $\int \frac{\tan x+1}{\tan x-1}dx = \int \frac{\sin x+\cos x}{\sin x-\cos x}dx = \log|\sin x-\cos x|+C$

(vii) $\displaystyle\int \frac{\sec x}{\log(\sec x + \tan x)}\, dx = \log|\log(\sec x + \tan x)| + C$

**Note :** $\dfrac{d}{dx}\log(\sec x + \tan x) = \sec x$

(viii) $\displaystyle\int \frac{x^2 - 1}{x(x^2+1)}\, dx = \int \frac{1 - \frac{1}{x^2}}{x + \frac{1}{x}}\, dx = \log\left|x + \frac{1}{x}\right| + C$

(ix) $\displaystyle\int \frac{dx}{x + \sqrt{x}} = \int \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)} = 2\int \frac{\frac{1}{2\sqrt{x}}}{\left(\sqrt{x}+1\right)}\, dx = 2\log|\sqrt{x}+1| + C$

(x) $\displaystyle\int \frac{x}{(x^4+1)\tan^{-1}x^2}\, dx = \frac{1}{2}\int \frac{\frac{2x}{1+x^4}}{\tan^{-1}x^2}\, dx = \frac{1}{2}\log|\tan^{-1}x^2| + C$

(xi) $\displaystyle\int \frac{dx}{x\log x \log\log x} = \int \frac{\frac{1}{x\log x}dx}{\log\log x} = \log|\log\log x| + C$

(xii) $\displaystyle\int \frac{\sin 2x}{1 + \sin^2 x}\, dx = \log|1 + \sin^2 x| + C$

(xiii) $\displaystyle\int \frac{e^{x-1} + x^{e-1}}{e^x + x^e}\, dx = \frac{1}{e}\int \frac{e^x + ex^{e-1}}{e^x + x^e}\, dx = \frac{1}{e}\log|e^x + x^e| + C$

## 1.6 Integration of some trigonometric Functions using Standard Results

When the integrand is a trigonometrical function, we use the trigonometrical formulae to transform the given function into standard integrands or their algebraic sum. Before proceeding further in this section your must revise all the trigonometric formulae given in Module - 1.

Based on Trigonometric formulae, study the following examples.

### *Illustrating the Concepts :*

*Evaluate :*

(i) $\int \sin^2 x\, dx$ (ii) $\int \sin^3 x\, dx$ (iii) $\int \sin^4 x\, dx$ (iv) $\int \sin^4 x \cos^4 x\, dx$

Hint : *Reduce the degree of integrand to one by transforming it into multiple angles of sine and cosine.*

(i) $\displaystyle\int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2}\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$

(ii) $\int \sin^3 x\, dx = \int \dfrac{3\sin x - \sin 3x}{4}\, dx$

$= \dfrac{1}{4}\left[\int 3\sin x\, dx - \int \sin 3x\, dx\right]$

$= \dfrac{1}{4}\left[-3\cos x + \dfrac{\cos 3x}{3}\right] + C$

$= \dfrac{-3}{4}\cos x + \dfrac{1}{12}\cos 3x + C$

(iii) $\int \sin^4 x\, dx = \int \left(\dfrac{1-\cos 2x}{2}\right)^2 dx$

$= \dfrac{1}{4}\int (1 - 2\cos 2x)\, dx + \dfrac{1}{4}\int (\cos^2 2x dx)$

$= \dfrac{x}{4} - \dfrac{1}{4}\sin 2x + \dfrac{1}{8}\int (1+\cos 4x)\, dx$

$= \dfrac{x}{4} - \dfrac{1}{4}\sin 2x + \dfrac{x}{8} + \dfrac{\sin 4x}{32} + C$

(iv) $\int \sin^4 x \cos^4 x\, dx = \dfrac{1}{16}\int \sin^4 2x$

Now proceed on the pattern of $\int \sin^4 x\, dx$ .

***Illustrating the Concepts*** **:**

*Evaluate :* (i) $\int \dfrac{dx}{1+\sin x}$ (ii) $\int \dfrac{dx}{1+\cos x}$.

(i) $\int \dfrac{dx}{1+\sin x} = \int \dfrac{1-\sin x}{\cos^2 x}\, dx$

$= \int \sec^2 x\, dx - \int \sec x \tan x\, dx$

$= \tan x - \sec x + C$

Alternative Method

$\int \dfrac{dx}{1+\sin x} = \int \dfrac{dx}{1+\cos\left(\dfrac{\pi}{2} - x\right)} = \int \dfrac{dx}{2\cos^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right)}$

$= \dfrac{1}{2}\int \sec^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right) dx = \dfrac{1}{2}\,\dfrac{\tan\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right)}{-1/2} + C$

$= -\tan\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right) + C$

(ii) $\int \dfrac{dx}{1+\cos x} = \int \dfrac{1-\cos x}{\sin^2 x}$

$= \int \operatorname{cosec}^2 x\, dx - \int \operatorname{cosec} x \cot x\, dx$

$= -\cot x + \operatorname{cosec} x + C$

Alternative Method

$$\int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2 x/2} = \frac{1}{2}\int \sec^2 \frac{x}{2}\, dx$$

$$= \frac{1}{2}\frac{\tan x/2}{1/2} + C = \tan\frac{x}{2} + C$$

***Illustrating the Concepts :***

*Evaluate the following integrals :*

(i) $\int \sin 2x \sin 3x\, dx$ (ii) $\int \sin 2x \cos 4x \cos 5x\, dx$ (iii) $\int \sin^2 x \cos^2 x\, dx$

Hint : *Apply trigonometric formulas to convert product form of sines and cosines into sum of sines and consines of multiple angle.*

(i) $\int \sin 2x \sin 3x\, dx = \frac{1}{2}\int 2\sin 2x \sin 3x\, dx$

$$= \frac{1}{2}\int (\cos x - \cos 5x)\, dx = \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C$$

(ii) $\int \sin 2x \cos 4x \cos 5x\, dx = \frac{1}{2}\int (2\sin 2x \cos 4x)\cos 5x\, dx$

$$= \frac{1}{2}\int (\sin 6x - \sin 2x)\ \cos 5x\, dx = \frac{1}{4}\int 2\sin 6x \cos 5x\, dx - \frac{1}{4}\int 2\sin 2x \cos 5x\, dx$$

$$= \frac{1}{4}\int (\sin 11x + \sin x)\, dx - \frac{1}{4}\int (\sin 7x - \sin 3x)\, dx$$

$$= -\frac{1}{4}\frac{\cos 11x}{11} - \frac{1}{4}\cos x + \frac{1}{4}\frac{\cos 7x}{7} - \frac{1}{4}\frac{\cos 3x}{3} + C$$

(iii) $\int \sin^2 x \cos^2 x\, dx = \frac{1}{4}\int \sin^2 2x\, dx = \frac{1}{4}\int \frac{1-\cos 4x}{2}\, dx = \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C$

**Illustration - 1** If : $\int \frac{dx}{a \sin x + b \cos x} = \frac{1}{k} \log |\operatorname{cosec}(x+\alpha) - \cot(x+\alpha)| + C$, *then* :

(A) $k = \sqrt{a^2+b^2},\ \alpha = \tan^{-1}\left(\frac{a}{b}\right)$

(B) $k = \sqrt{a^2+b^2},\ \alpha = \tan^{-1}\left(\frac{b}{a}\right)$

(C) $k = \sqrt{a^2b^2},\ \alpha = \tan^{-1}\left(\frac{a}{b}\right)$

(D) $k = \sqrt{a^2+b^2},\ \alpha = \tan^{-1}\left(\frac{b}{a}\right)$

**SOLUTION : (B)**

$$\int \frac{dx}{a \sin x + b \cos x} = \frac{1}{\sqrt{a^2+b^2}} \int \frac{dx}{\frac{a}{\sqrt{a^2+b^2}} \sin x + \frac{b}{\sqrt{a^2+b^2}} \cos x}$$

$$= \frac{1}{\sqrt{a^2+b^2}} \int \frac{dx}{\sin x \cos \alpha + \cos x \sin \alpha}$$ where $\alpha = \tan^{-1}(b/a)$

$$= \frac{1}{\sqrt{a^2+b^2}} \int \operatorname{cosec}(x+\alpha)\, dx$$

$$= \frac{1}{\sqrt{a^2+b^2}} \log |\operatorname{cosec}(x+\alpha) - \cot(x+\alpha)| + C$$ where $\alpha = \tan^{-1}(b/a)$

## METHOD OF INTEGRATION

We have the following methods of Integration :

(a) Integration by Substitution (or change of independent variable)

(b) Integration by Parts

(c) Integration by Partial Fraction

## IN-CHAPTER EXERCISE - A

*Evaluate the following integrals.*

(i) $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$

(ii) $\int\frac{x^3+3x^2+4}{\sqrt{x}}dx$

(iii) $\int\frac{dx}{\sqrt{ax+b}-\sqrt{ax+c}}$

(iv) $\int\frac{x^3+3x^2+2x+1}{x-1}dx$

(v) $\int cos^3\ x\ dx$

(vi) $\int(tan\ x+cot\ x)^2\ dx$

(vii) $\int sin\ x\sqrt{1-\cos 2x}\ dx$

(viii) $\int\left(e^{a\ log\ x}+e^{x\ log\ a}\right)dx$

(ix) $\int cos\ x\ cos\ 2x\ cos3x\ dx$

(x) $\int(1+x)\sqrt{1-x}\ dx$

(xi) $\int\frac{2x-3}{\sqrt{2x+1}}dx$

(xii) $\int\frac{sin\ x+cos\ x}{\sqrt{1+sin\ 2x}}dx$

(xiii) $\int sin\ x\ cos\ x\ (sin2x+cos2x)\ dx$

(xiv) $\int\frac{sin^2\ x}{(1+cos\ x)^2}dx$

(xv) $\int\frac{1}{1-sin\ x}dx$

(xvi) $\int\frac{1}{1-cosx}dx$

## INTEGRATION BY SUBSTITUTION Section - 2

### 2.1 Introduction

If $g(x)$ is a continuous and differentiable function, then to evaluate the integral of the form $\int f[g(x)]\, g'(x)\, dx$, we substitute $g(x) = t$ and $g'(x)\, dx = dt$.

The substitution reduces the integral $\int f[g(x)]\, g'(x)\, dx$ to the form $\int f(t)\, dt$ i.e.

We have changed the variable of integration form $x$ to $t$. This substitution process can reduce the integrand to the form where we can evaluate the integral by using the standard results or it can reduce the complexity of the integrand. After evaluating this integral we substitute back the value of $t$.

**Illustration - 2** $\int e^x (x+1) \cos(x e^x)\, dx =$

**(A)** $\sin(xe^x)$ **(B)** $\cos(xe^x)$ **(C)** $e^x \sin x$ **(D)** $e^x \cos x$

**SOLUTION : (A)**

The given integral is in terms of the variable $x$, we can simplify the integral by connecting it in the terms of another variable $t$ using substitution.

Here let us put $xe^x = t$

and hence $xe^x\, dx + e^x dx = dt$

$\Rightarrow \quad e^x (x + 1)\, dx = dt$

The given integral $= \int \cos(xe^x)\, [e^x (x+1)\, dx]$

$= \int \cos t\, dt = \sin t + C$

$= \sin(x e^x) + C$

***Note :*** The final result of the problem must be in terms of $x$.

***Illustrating the Concepts :***

*Evaluate :* (i) $\int \frac{x^2}{1+x^6}\, dx$ (ii) $\int \frac{dx}{x + \sqrt[3]{x}}$ (iii) $\int \frac{x^2}{\sqrt{1+x}}\, dx$

(i) Let $x^3 = t \quad \Rightarrow \quad 3x^2\, dx = dt$

$$\Rightarrow \int \frac{x^2 dx}{1+x^6} = \frac{1}{3}\int \frac{3x^2 dx}{1+x^6} = \frac{1}{3}\int \frac{dt}{1+t^2}$$

$$= \frac{1}{3}\tan^{-1} t + C = \frac{1}{3}\tan^{-1} x^3 + C$$

(ii) $\sqrt[3]{x}$ indicates that we should try $x = t^3$

$\Rightarrow \quad dx = 3t^2\, dt$

$$\Rightarrow \int \frac{dx}{x + \sqrt[3]{x}} = \int \frac{3t^2 dt}{t^3 + t} = 3\int \frac{t dt}{t^2+1} = \frac{3}{2}\int \frac{2t\, dt}{t^2+1}$$

$$= \frac{3}{2}\log|t^2+1| + C = \frac{3}{2}\log|x^{2/3}+1| + C$$

(iii) Let $1 + x = t^2 \quad \Rightarrow \quad dx = 2t\, dt$

$$\Rightarrow \int \frac{(t^2-1)^2}{t}\, 2t\, dt = 2\int (t^4 - 2t^2 + 1)\, dt$$

$$= 2\frac{t^5}{5} + 2t - \frac{4t^3}{3} + C$$

$$= \frac{2}{3}(1+x)^{5/2} + 2\sqrt{1+x} - \frac{4}{3}(1+x)^{3/2} + C$$

***Illustrating the Concepts :***

*Evaluate :* (i) $\int \frac{a^{x+\tan^{-1} a^x}}{a^{2x}+1}\,dx$ (ii) $\int \sqrt{\sin\theta}\cos^3\theta\,d\theta$ (iii) $\int \sqrt{\frac{x}{1-x^3}}\,dx$

(i) The given integral can be written as :

$$\int \frac{a^x a^{\tan^{-1} a^x}}{a^{2x}+1}\,dx$$

Let $\tan^{-1} a^x = t$

$$\Rightarrow \quad \frac{1}{1+a^{2x}} a^x \log a\; dx = dt$$

$$\Rightarrow \quad I = \int \frac{a^{\tan^{-1} a^x}\, a^x \log a\; dx}{\log a\,(1+a^{2x})} = \int \frac{a^t\,dt}{\log a}$$

$$\Rightarrow \quad I = \frac{1}{\log a}\frac{a^t}{\log a} + C$$

$$\Rightarrow \quad I = \frac{a^{\tan^{-1} a^x}}{(\log a)^2} + C$$

(ii) Let $\sin\theta = t^2 \quad \Rightarrow \quad \cos\theta\, d\theta = 2t\,dt$

$\Rightarrow$

$$\int \sqrt{\sin\theta}\,\cos^3\theta\,d\theta = \int \sqrt{\sin\theta}\,(1-\sin^2\theta)\cos\theta\,d\theta$$

$$= \int t\,(1-t^4)\,2t\,dt$$

$$= \frac{2t^3}{3} - \frac{2t^7}{7} + C = \frac{2}{3}(\sin\theta)^{3/2} - \frac{2}{7}(\sin\theta)^{7/2} + C$$

(iii) The given integral is

$$I = \int \frac{\sqrt{x}\,dx}{\sqrt{1-x^3}}$$

$\sqrt{x}$ appears in the derivative of $x^{3/2}$

hence, let $x^{3/2} = t \quad \Rightarrow \quad 3/2\,\sqrt{x}\,dx = dt$

$$\Rightarrow \quad I = \frac{2}{3}\int \frac{\frac{3}{2}\sqrt{x}\,dx}{\sqrt{1-x^3}} = \frac{2}{3}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{2}{3}\sin^{-1} t + C = \frac{2}{3}\sin^{-1} x^{3/2} + C$$

***Illustrating the Concepts :***

*Evaluate the following integrals :*

(i) $\int \tan^2 x\,dx$ (ii) $\int \tan^3 x\,dx$ (iii) $\int \tan^4 x\,dx$ (iv) $\int \sec^4 x\,dx$

(i) $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

(ii) $\int \tan^3 x\,dx = \int \tan x\,(\sec^2 x - 1)\,dx$

$$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$$

$$= \frac{\tan^2 x}{2} - \log|\sec x| + C$$

(iii) $\int \tan^4 x\,dx \int \tan^2 x\,(\sec^2 c - 1)\,dx$

$$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$$

$$= \frac{\tan^3 x}{3} - \tan x + x + C \qquad \text{[from (i)]}$$

(iv) $\int \sec^4 x\,dx = \int \sec^2 \sec^2 x\,dx$

$$= \int (1+\tan^2 x)\sec^2 x\,dx$$

$$= \int (1+t^2)\, dt = t + t^3/3 + C$$

$$= \tan x + \frac{\tan^3 x}{3} + C$$

## General Procedure :

When the integrand is of the form $\tan^n x \sec^n x$, the following cases arise :

(a) When $m$ is even positive integer, substitute $\tan x = z$.

(b) When $n$ is odd positive integer, substitute $\sec x = z$.

When the integrand is of the form $\cot^n x \ \mathrm{cosec}^m x$, the following cases arise :

(a) When $m$ is even positive integer, substitute $\cot x = z$.

(b) When $n$ is odd positive integer, substitute cosec $= z$.

### *Illustrating the Concepts :*

*Evaluate :* (i) $\int \sin^3 x \cos^4 x\, dx$ (ii) $\int \sin^5 x\, dx$

(i) $\int \sin^3 x \cos^4 x\, dx = \int \sin^2 x \ \cos^4 x\,(\sin x\, dx)$

$$= -\int (1-t^2)\, t^4\, dt \quad \text{where } t = \cos x$$

$$= \frac{t^7}{7} - \frac{t^5}{5} + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$$

(ii) $\int \sin^5 x\, dx = \int \sin^4 x \sin x\, dx$

$$= -\int (1-\cos^2 x)^2 (-\sin x\, dx)$$

$$= -\int (1-t^2)^2\, dt \quad \text{where } t = \cos x$$

$$= -\int (1+t^4-2t^2)\, dt = -t - \frac{t^5}{5} + \frac{2t^3}{3} + C$$

$$= -\cos x - \frac{\cos^5 x}{5} + \frac{2}{3}\cos^3 x + C$$

## General Procedure :

When the integrand is of the form $\sin^m x \cos^n x$, the following cases arise :

(a) When $m$ is odd positive integer, substitute $\cos x = z$.

(b) When $n$ is odd positive integer, substitute $\sin x = z$.

## IN-CHAPTER EXERCISE - B

*Evaluate the following Integrals.*

(i) $\displaystyle\int \frac{dx}{e^x + e^{-x}}$

(ii) $\displaystyle\int \frac{x^5}{(1-x^3)^2}\,dx$

(iii) $\displaystyle\int \frac{x^3}{\sqrt{x^8+1}}\,dx$

(iv) $\displaystyle\int \frac{2x\tan^{-1}x^2}{1+x^4}\,dx$

(v) $\displaystyle\int \frac{\sin x + \cos x}{(\sin x - \cos x)^3}\,dx$

(vi) $\displaystyle\int \frac{x\,dx}{(ax^2+b)^p}$

(vii) $\displaystyle\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$

(viii) $\displaystyle\int \cos x \tan^3 x\,dx$

(ix) $\displaystyle\int \tan x \sec^6 x\,dx$

(x) $\displaystyle\int \tan^3 x \sec^5 x\,dx$

(xi) $\displaystyle\int \frac{dx}{x\cos^2(\log x)}$

(xii) $\displaystyle\int \frac{x^3 dx}{(x^2-a^2)^{3/2}}$

## 2.2 Some Important Substitutions

Following are some important substitutions useful in evaluating integrals.

| Expression | Substitution |
|---|---|
| $a^2 + x^2$ | $x = a\tan\theta$ or $x = a\cot\theta$ |
| $a^2 - x^2$ | $x = a\sin\theta$ or $x = a\cos\theta$ |
| $x^2 - a^2$ | $x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$ |
| $\sqrt{\dfrac{a-x}{a+x}}$ or $\sqrt{\dfrac{a+x}{a-x}}$ | $x = a\cos 2\theta$ |
| $\sqrt{\dfrac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$ | $x = \alpha\cos^2\theta + \beta\sin^2\theta$ |

## 2.3 Some standard results derived using substitution method

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

[use $x = a\tan\theta$ or $x = a\cot\theta$ to derive the result]

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x+a}{x-a}\right| + C$$

[use $x = a\sec\theta$ or $x = \operatorname{cosec}\theta$ to derive the result]

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ell og\left|\frac{a+x}{a-x}\right| + C$$

[use $x = a\cos\theta$ or $x = a\sin\theta$ to derive the result]

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$$

[use $x = a\tan\theta$ or $x = \cot\theta$ to derive the result]

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$$

[use $x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$ to derive the result]

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

[use $= x$ a sin $\theta$ or $x =$ a cos $\theta$ to derive the result]

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

[use $x = a\sec\theta$ or $x = a\,cosec\,\theta$ to derive the result]

$$\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right| + C$$

[Use $x = \tan\theta$ or $x = \cot\theta$ to drive the result]

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$ [Use $x = sec\,\theta$ or $x = cosec\,\theta$ to derive the result]

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$ [Use $x = sin\,\theta$ or $x = cos\,\theta$ to derive the result]

## 2.4 Evaluation of Integrals of various types based on the formulas given in section 2.3

### 2.4.1 Type-I

Integrals of the type $\int \frac{1}{ax^2 + bx + c}dx, \int \frac{1}{\sqrt{ax^2\ bx + c}}dx, \int \sqrt{ax^2 + bx + c\ dx}$.

To evaluate this type of integrals we express quadratic expression $ax^2 + bx + c$ in the form of perfect square and then apply the formulas given in section 2.3.

TYPE : $\int \frac{dx}{ax^2 + bx + 2}$

***Illustrating the Concepts :***

*Evaluate* : (i) $\int \frac{dx}{x^2 + x + 1}$ (ii) $\int \frac{dx}{1 - 4x - 2x^2}$ (iii) $\int \frac{dx}{x^2 + 6x + 1}$

(i) $$\int \frac{dx}{x^2 + x + 1} = \int \frac{dx}{x^2 + 2\frac{1}{2}x + \frac{3}{4} + \frac{1}{4}}$$

$$= \int \frac{dx}{\left(a + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x + 1/2}{\sqrt{3}/2}\right) + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

(ii) $$\int \frac{dx}{1 - 4x - 2x^2} = \frac{1}{2}\int \frac{dx}{1/2 - (x^2 + 2x)}$$

$$= \frac{1}{2}\int \frac{dx}{\left(\sqrt{3/2}\right)^2 - (x+1)^2}$$

$$= \frac{1}{2}\,\frac{1}{2\sqrt{3/2}}\log\left|\frac{\sqrt{3/2} + x + 1}{\sqrt{3/2}\ \ (x+1)}\right| + C$$

(iii) $$\int \frac{dx}{x^2 + 6x + 1} = \int \frac{dx}{x^2 + 6x + 9 - 8}$$

$$= \int \frac{dx}{(x+3)^2 - (2\sqrt{2})^2}$$

$$= \frac{1}{2(2\sqrt{2})}\log\left|\frac{x + 3 - 2\sqrt{2}}{x + 3 + 2\sqrt{2}}\right| + C$$

TYPE : $\int \frac{dx}{\sqrt{ax^2 + bx + C}}$

***Illustrating the Concepts :***

*Evaluate* : (i) $I = \int \frac{dx}{\sqrt{1 - x - x^2}}$ (ii) $I = \int \frac{dx}{\sqrt{2x^2 + 6x + 2}}$

(i) Let $I = \int \frac{dx}{\sqrt{1 - x - x^2}}$

Treat $1 - x - x^2$ as $1 - (x + x^2)$

$= 5/4 - (x^2 + x + 1/4) = 5/4 - (x + 1/2)^2$

$$\Rightarrow \quad I = \int \frac{dx}{\sqrt{\frac{5}{4} - \left(x + \frac{1}{2}\right)^2}} = \sin^{-1}\left(\frac{x + 1/2}{\sqrt{5/2}}\right)$$

$$= \sin^{-1}\left(\frac{2x + 1}{\sqrt{5}}\right) + C$$

(ii) Let $I = \int \frac{dx}{\sqrt{2x^2 + 6x + 2}}$

Now $2x^2 + 6x + 2 = 2(x^2 + 3x + 1)$

$$= 2\left(x^2 + \frac{6x}{2} + \frac{9}{4} - \frac{9}{4} + 1\right) = 2\left[\left(x + \frac{3}{2}\right)^2 - \frac{5}{4}\right]$$

This is in the form $x^2 - a^2$.

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x + 3/2)^2 - 5/4}}$$

$$= \frac{1}{\sqrt{2}} \log\left| x + \frac{3}{2} + \sqrt{(x + 3/2)^2 - 5/4} \right| + C$$

TYPE : $\int \sqrt{ax^2 + bx + c}\, dx$

## Illustration - 3

If $I = \int \sqrt{x^2 + 5x + 1}\, dx = \frac{2x + 5}{4}\sqrt{x^2 + 5x + 1} + A \log\left| x + \frac{5}{2} + \sqrt{x^2 + 5x + 1} \right| + C$, then A =

(A) $\frac{21}{4}$ (B) $\frac{-21}{4}$ (C) $\frac{21}{8}$ (D) $\frac{-21}{8}$

**SOLUTION : (D)**

Let $I = \int \sqrt{x^2 + 5x + 1}\, dx = \int \sqrt{(x + 5/2)^2 - 21/4}\, dx$

$$= \frac{x + 5/2}{2}\sqrt{x^2 + 5x + 1} - \frac{21}{8}\log\left| x + 5/2 + \sqrt{(x + 5/2)^2 - 21/4} \right| + C$$

$$= \frac{2x + 5}{4}\sqrt{x^2 + 5x + 1} - \frac{21}{8}\log\left| x + 5/2 + \sqrt{x^2 + 5x + 1} \right| + C$$

### 2.4.2 Type-II

Integrals of the type :

(a) $\int \frac{px+q}{ax^2+bx+c}dx,$ (b) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx,$ (c) $\int \frac{p(x)}{ax^2\ bx+c}dx$

where $p(x)$ is a polynomial of degree greater than or equal to 2.

To evaluate the integrals of type $(a)$ and $(b)$ we express the linear factor in the numerator as :

$px + q = m$ (derivative of quadratic expression in the denominator) $+ n$

(where $m$ and $n$ are unknown constant to be determined by equating the coefficients of $x$ and constant terms on both sides)

$\Rightarrow \quad px + q = m(2ax + b) + n$

$\Rightarrow \quad 2am = p$ **. . . (i)** and $mb + n = q$ **. . . (ii)**

Solving (i) and (ii), we get $m = p/2a$ and $n = q - bp/2a$

Consider type (a)

$$\int \frac{px+q}{ax^2+bx+c}dx = \frac{p}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx + \left(q-\frac{bp}{2a}\right)\int \frac{1}{ax^2+bx+c}dx$$

$$= \frac{P}{2a}\log\left|ax^2+bx+c\right| + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{ax^2+bx+c}$$

The other integral on RHS can be evaluated with the help of methods discussed in section 2.41

Consider type (b)

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx = \frac{p}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx + \left(q-\frac{bp}{2a}\right)\int \frac{1}{\sqrt{ax^2+bx+c}}dx$$

$$\Rightarrow \quad \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx = \frac{p}{a}\sqrt{ax^2+bx+c} + \left(q-\frac{bp}{2a}\right)\int \frac{1}{\sqrt{ax2+bx+c}}dx$$

The other integral on RHS can be evaluated with the help of methods discussed in section 2.4.1.

Consider type (c)

In this case divide the numerator by the denominator and express the integrand as :

$$Q(x) + \frac{R(x)}{ax^2+bx+c},$$ where $R(x)$ is a linear function of $x$.

Therefore $\int \frac{P(x)}{ax^2+bx+c}dx = \int Q(x)dx + \int \frac{R(x)}{ax^2+bx+c}dx$

Now to evaluate the integral on RHS apply the method discussed in part (a).

## Illustration - 4

*If* $\int \frac{3x+1}{\sqrt{x^2+4x+1}} = A\sqrt{x^2+4x+1} + B\log\left|x+2+\sqrt{x^2+4x+1}\right| + C,$ *then* :

**(A)** A = 3, B = 5 **(B)** $A=-3, B=5$ **(C)** $A=3, B=-5$ **(D)** $A=-3, B=-5$

**SOLUTION : (C)**

The linear expression in the numerator can be expressed as :

$$3x+1 = l\left(\frac{d}{dx}(x^2+4x+1)\right)+m$$

$$\Rightarrow \; 3x+1 = l\,(2x+4)+m$$

Comparing the coefficient of $x$ and $x^0$,

$3 = 2l$ and $l = 4l + m$

$\Rightarrow$ $l = 3/2$ and $m = -5$

$$\Rightarrow I = \int \frac{3x+1}{\sqrt{x^2+4x+1}} = \int \frac{3/2\,(2x+4)-5}{\sqrt{x^2+4x+1}}dx$$

$$= \frac{3}{2}\int \frac{2x+4\,dx}{\sqrt{x^2+4x+1}} - 5\int \frac{dx}{\sqrt{x^2+4x+1}}$$

Let $I_1 = \frac{3}{2}\int \frac{2x+4}{\sqrt{x^2+4x+1}} = \frac{3}{2}\int \frac{dt}{\sqrt{t}}$

(where $t = x^2 + 4x + 1$)

$$= 3\sqrt{t} + C = 3\sqrt{x^2+4x+1} + C$$

Let $I_2 = 5\int \frac{dx}{\sqrt{x^2+4x+1}} = 5\int \frac{dx}{\sqrt{(x+2)^2-3}}$

$$= 5\log\left|x+2+\sqrt{(x+2)^2-3}\right| + C$$

$$\Rightarrow I = I_1 - I_2$$

$$= 3\sqrt{x^2+4x+1} - 5\log\left|x+2+\sqrt{x^2+4x+1}\right| + C$$

## Illustration - 5

*If* $\int \frac{x^2-x+1}{2x^2+x+2}dx = \frac{x}{2} + A\log\left|2x^2+x+2\right| + B\tan^{-1}\left(\frac{4x+1}{\sqrt{15}}\right) + C,$ *then* :

**(A)** $A = \frac{3}{8}, B = \frac{3}{4\sqrt{15}}$ **(B)** $A = \frac{3}{8}, B = \frac{-3}{4\sqrt{15}}$

**(C)** $A = \frac{-3}{8}, B = \frac{3}{4\sqrt{15}}$ **(D)** $A = \frac{-3}{8}, B = \frac{-3}{4\sqrt{15}}$

**SOLUTION : (C)**

Express numerator in terms of denominator and its derivative.

Let $x^2 - x + 1 = l\,(2x^2 + x + 2) + m\,(4x + 1) + n$

$\Rightarrow 1 = 2l, \; -1 = l + 4m, \; 1 = 2l + m + n,$

$\Rightarrow \; \ell = 1/2, \; m = -3/8, \; n = 3/8$

$$\Rightarrow I = \int \frac{x^2-x+1}{2x^2+x+2}dx$$

$$= \int \frac{1/2\,(2x^2+x+2) - 3/8\,(4x+1) + 3/8}{2x^2+x+2}dx$$

$$\Rightarrow \frac{1}{2}\int dx - \frac{3}{8}\int \frac{4x+1}{2x^2+x+2}dx + \frac{3}{8}\int \frac{dx}{2x^2+x+2}$$

$$= \frac{x}{2} - \frac{3}{8}\log\left|2x^2+x+2\right| + \frac{3}{8}I_1$$

where $I_1 = \int \frac{dx}{2x^2 + x + 2}$

$$= \frac{1}{2}\int \frac{dx}{x^2 + 1/2x + 1/16 - 1/16 + 1} + \frac{1}{2}\int \frac{dx}{(x+1/14)^2 + 15/16}$$

$$= \frac{1}{2}\frac{1}{\sqrt{15}/4}\tan^{-1}\left(\frac{x+1/4}{\sqrt{15}/4}\right) + C = \frac{2}{\sqrt{15}}\tan^{-1}\left(\frac{4x+1}{\sqrt{15}}\right) + C$$

$$\Rightarrow I = \frac{x}{2} - \frac{8}{3}\log|2x^2 + x + 2| + \frac{3}{4\sqrt{15}}\tan^{-1}\left(\frac{4x+1}{\sqrt{15}}\right) + C$$

### 2.4.3 Type-III

Integrals of the type

$$\int \frac{dx}{a\sin^2 x + b\cos^2 x}, \int \frac{dx}{a + b\sin^2 x}, \int \frac{dx}{a + b\cos x},$$

$$\int \frac{dx}{(a\sin x + b\cos x)^2}, \int \frac{f(\tan x)\,dx}{a\,b\sin^2 x + \cos^2 x + d\sin x\cos x}$$ where $f(\tan x)$ is a polynomial in $\tan x$.

To evaluate this type of integrals we proceed in the following manner :

STEP - I Divide numerator and denominator both by $\cos^2 x$.

STEP - II Put $\tan x = t$, $\sec^2 x\, dx = dt$.

This substitution will convert the trigonometric integral into algebraic integral.

After employing these steps the integral will reduce to the form $\int \frac{f(t)dt}{At^2 + B + C}$,

where $f(t)$ is a polynomial in $t$.

This integral can be evaluated by methods discussed in section 2.41 and 2.42.

**Illustration - 6**

$$\int \frac{dx}{3\,sin^2 x + 4\,cos^2 x} =$$

**(A)** $\frac{1}{2/\sqrt{3}}\tan^{-1}\left(\frac{\tan x}{2/\sqrt{3}}\right)$ **(B)** $\frac{1}{6/\sqrt{3}}\tan^{-1}\left(\frac{\tan x}{2/\sqrt{3}}\right)$

**(C)** $\frac{1}{2/\sqrt{3}}\tan^{-1}\left(\frac{\tan x}{6/\sqrt{3}}\right)$ **(D)** $\frac{1}{6/\sqrt{3}}\tan^{-1}\left(\frac{\tan x}{6/\sqrt{3}}\right)$

**SOLUTION : (B)**

$$\int \frac{dx}{3\sin^2 x + 4\cos^2 x} = \int \frac{\sec^2 x}{3\tan^2 x + 4}\, dx$$

$$= \frac{1}{3}\int \frac{dt}{t^2 + (2\sqrt{3})^2} = \frac{1}{3}\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{t}{2\sqrt{3}}\right) + C \qquad \text{where } \tan x = t$$

$$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{2}\tan x\right) + C$$

## 2.4.4 Type-IV

Integrals of the type $\int \frac{dx}{a\sin x + b\cos x}, \int \frac{dx}{a + b\sin x}, \int \frac{dx}{a + b\cos x}, \int \frac{f\left(\tan \frac{x}{2}\right) dx}{a\sin x + b\cos x + c}$

where $f(\tan x/2)$ is a polynomial in $\tan x/2$.
To evaluate this type of integrals we proceed in the following manner :

**STEP - I** Put $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

**STEP - II** Replace $1 + \tan^2 x/2$ in the numerator by $\sec^2 x/2$.

**STEP - III** Put $\tan x/2 = t$ and $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$

After performing these three steps the integral reduces to the form $\int \frac{f(t)\, dt}{At^2 + BT + C}$ where $f(t)$ is a polyno -mial in $t$. This integral can be evaluated by methods discussed in section 2.41 and 2.42.

### *Illustrating the Concepts* :

*Evaluate* : (i) $\int \frac{dx}{4 + 5\sin x}$ (ii) $\int \frac{dx}{a + b\cos x}$ *where* $a, b > 0$.

(i) Let $I = \int \frac{dx}{4 + 5\sin x}$

Put $\tan\frac{x}{2} = t \Rightarrow x = 2\tan^{-1} t \Rightarrow \cos x = \frac{1-t^2}{1+t^2}$ ; $\sin x = \frac{2t}{1+t^2}$ ; $dx = \frac{2dt}{1+t^2}$

$$\Rightarrow \quad I = \int \frac{\frac{2dt}{1+t^2}}{4+5\left(\frac{2t}{1+t^2}\right)} = \int \frac{2dt}{4t^2+10t+4}$$

$$= \frac{1}{2}\int \frac{dt}{t^2+5t/2+1}$$

$$= \frac{1}{2}\int \frac{dt}{(t+5/4)^2-9/16}$$

$$= \frac{1}{2}\,\frac{1}{2\times 3/4}\log\left|\frac{t+5/4-3/4}{t+5/4+3/4}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{4t+2}{4t+8}\right| + C = \frac{1}{3}\log\left|\frac{2\tan\frac{x}{2}+1}{2\tan\frac{x}{2}+4}\right| + C$$

(ii) $\displaystyle\int \frac{dx}{a+b\,coc\,x}$ where $a, b > 0$.

Let $\quad \tan x/2 = t$

$$\Rightarrow \quad I = \int \frac{dx}{a+b\cos x}\int \frac{\frac{2dt}{1+t^2}}{a+b\left(\frac{1-t^2}{1+t^2}\right)}$$

$$\Rightarrow \quad \int \frac{2dt}{(a+b)+(a-b)\,t^2}$$

Case 1 : Let $a = b$

$$\Rightarrow \quad I = \int \frac{2dt}{a+b} = \frac{2t}{a+b} = \frac{2}{a+b}\tan\frac{x}{2} + C$$

Case 2 : Let $a > b$

$$\Rightarrow \quad I = \frac{1}{a-b}\int \frac{2dt}{\frac{a+b}{a-b}+t^2}$$

$$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\tan^{-1}\left(t\sqrt{\frac{a-b}{a+b}}\right) + C$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\tan\frac{x}{2}\sqrt{\frac{a-b}{a+b}}\right) + C$$

Case 3 : Let $a < b$

$$I = \int \frac{2dt}{(a+b)-(b-a)t^2} = \frac{2}{b-a}\int \frac{dt}{\frac{b+a}{b-a}-t^2}$$

$$= \frac{2}{b-a}\,\frac{\sqrt{b-a}}{2\sqrt{b+a}}\log\left|\frac{\sqrt{\frac{b+a}{b-a}}+t}{\sqrt{\frac{b+a}{b-a}}-t}\right| + C$$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b+a}+\sqrt{b-a}\,\tan\frac{x}{2}}{b+a-\sqrt{b-a}\,\tan\frac{x}{2}}\right| + C$$

## 2.4.5 Type-V

Integrals of the type $\displaystyle\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$.

To evaluate this type of integrals express numerator as follows :

Numerator = $m$ (Derivative of Denominator) + $n$ (denominator)

$\Rightarrow \quad (a \sin x + b \cos x) = m\,(c \cos x - d \sin x) + n\,(c \sin x + d \cos x)$

where $m$ and $n$ are constants to be determined by comparing the coefficients of sin$x$ and cos$x$ on both sides.

Therefore,

$$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx = \int \frac{m\,(c\cos x - d\sin x) + n\,(c\sin x + d\cos x)}{c\sin x + d\cos x}\,dx$$

$$= m\int \frac{\frac{d}{dx}(c\sin x + d\cos x)}{c\sin x + d\cos x}\,dx + n\int dx$$

$$= m\ln|c\sin x + d\cos x| + nx + C$$

**Illustration - 7**

If $\int \frac{2\sin x + 3\cos x}{\sin x + 4\cos x}\,dx = ax + b\log\left|\sin x + 4\cos x\right| + C$, then $a + b =$

(A) $\frac{9}{17}$ (B) $\frac{-9}{17}$ (C) $\frac{19}{17}$ (D) $\frac{-19}{17}$

**SOLUTION : (A)**

Let $2\sin x + 3\cos x = l(\sin x + 4\cos x) + m(\cos x - 4\sin x)$

Comparing coefficients of $\sin x$ and $\cos x$ :

$2 = l - 4m$, $3 = 4l + m$

$\Rightarrow l = 14/17$ $m = -5/17$

$$\Rightarrow \quad I = \int \frac{2\sin x + 3\cos x}{\sin x + 4\cos x}\,dx$$

$$\Rightarrow \quad I = \frac{14}{17}\int \frac{\sin x + 4\cos x}{\sin x + 4\cos x}\,dx - \frac{5}{17}\int \frac{\cos x - 4\sin x}{\sin x + 4\cos x}\,dx$$

$$\Rightarrow \quad I = \frac{14}{17}x - \frac{5}{17}\log|\sin x + 4\cos x| + C$$

### 2.4.6 Type - VI

Integrals of the type $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx$.

To evaluate this type of integrals express numerator as follows :

Numerator = $l$ (Derivative of Denominator) + $m$ (denominator) + $n$

$\Rightarrow$ $a\sin x + b\cos x + c = l(c\cos x - d\sin x) + m(c\sin x + d\cos x) + n$

where $l$, $m$ and $n$ are constants to be determined by comparing the coefficients of $\sin x$, $\cos x$ and constant terms on both sides.

Therefore,

$$\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx = \int \frac{l(p\cos x - q\sin x + r) + m(p\sin x + q\cos x + r) + n}{p\sin x + q\cos x + r}\,dx$$

$$= l\int \frac{\frac{d}{dx}(p\sin x + q\cos x + r)}{p\sin x + q\cos x + r}\,dx + m\int dx + n\int \frac{dx}{p\sin x + q\cos x + r}$$

$$= l\ln|p\sin x + q\cos x + r| + nx + n\int \frac{dx}{p\sin x + q\cos x + r} + C$$

The last integral on RHS can be evaluated by the methods given in Section. 2.44.

## 2.5 Special Type of Integrals

The following examples cover special type of algebraic integrals in which denominator is quadratic in $x^2$.

***Illustrating the Concepts :***

*Evaluate* : (i) $\int \frac{x^2+1}{x^4+1}dx$ (ii) $\int \frac{x^2-1}{x^4+1}dx$

(i) Let

$$I_1 = \int \frac{x^2+1}{x^4+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx$$

$$= \int \frac{dt}{t^2+2} \qquad \text{where } t = x - \frac{1}{x}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) + C$$

(ii) Let

$$I_2 = \int \frac{x^2-1}{x^4+1}dx = \int \frac{1-\frac{1}{x^2}}{\left(x^2+\frac{1}{x^2}\right)}dx$$

$$= \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}dx$$

Let $\quad I_2 = \int \frac{dt}{t^2-2} \quad$ where $t = x + \frac{1}{x}$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right| + C$$

**Illustration - 8**

*If* $\int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}dx = \frac{1}{\sqrt{2}} f\left(\frac{x^2+1}{x\sqrt{2}}\right) + C$, *then* $f(x) =$

(A) $\sin^{-1}|x|$ (B) $\tan^{-1}|x|$ (C) $\sec^{-1}|x|$ (D) $\log|x|$

**SOLUTION : (C)**

The given integral is

$$I = \int \frac{1-\frac{1}{x^2}}{\frac{x^2+1}{x}\sqrt{\frac{x^4+1}{x^2}}}dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}}}dx = \int \frac{dt}{t\sqrt{t^2-2}} \qquad \text{where } x + \frac{1}{x} = t.$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \sec^{-1}\left|\frac{t}{\sqrt{2}}\right| + C = \frac{1}{\sqrt{2}} \sec^{-1}\left|\frac{x^2+1}{x\sqrt{2}}\right| + C$$

## IN-CHAPTER EXERCISE - C

*Evaluate the following Integrals.*

(i) $\int \frac{x}{x^2+x+1} dx$

(ii) $\int (3x-2)\sqrt{x^2+4x+1}\, dx$

(iii) $\int (x+1)\sqrt{x^2+x+1}\, dx$

(iv) $\int \frac{x^2+x+1}{2x^2+4x+1} dx$

(v) $\int \frac{dx}{4\sin^2 x + 4\sin x\cos x + 5\cos^2 x}$

(vi) $\int \frac{dx}{4+5\cos x}$

(vii) $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x} dx$

(viii) $\int \frac{2+3\cos x}{\sin x + 2\cos x + 3} dx$

(ix) $\int \frac{dx}{(2\sin x + 3\cos x)^2}$

(x) $\int \frac{dx}{\cos x\,(\sin x + 2\cos x)}$

(xi) $\int \frac{x^2}{x^4+1} dx$

(xii) $\int \frac{dx}{x^4+1}$

(xiii) $\int \sqrt{\tan x}\, dx$

(xiv) $\int \sqrt{\cot x}\, dx$

(xv) $\int \frac{dx}{\sin^4 x + \cos^4 x}$

## INTEGRATION BY PARTS Section - 3

When integrand involves more than one type of functions we may solve them by the use of a rule which is known as integration by Parts. The rule, its derivation and uses are given below :

### 3.1 Derivation of the result

We known that $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\Rightarrow$ $d\,(uv) = udv + vdu$ $\Rightarrow$ $\int d(uv) = \int udv + \int vdu$ $\Rightarrow$ $\int udv = uv - \int vdu$

If we take $u$ as part 1 and $dv$ as part 2, then the above result can be written as :

$$\int (\text{part 1})\,(\text{part 2}) = (\text{part 1})\,[\text{integral of part 2}] - \int [\text{integral of part 2}][\text{differential of part 1}]$$

### 3.2 Proper choice of first and second functions

Integration with the help of above rule is called Integration by Parts. In the above rule there are two terms on RHS and in both in terms integral on the second function is involved. Therefore in the product of two functions if one of the two functions is not directly integrable (e.g. $\log x$, $\sin^{-1}x$, $\cos^{-1}$, $\tan^{-1}x$ etc.) we take it as the first function and the remaining function is taken as the second function. If there is no other function, then unity is taken as the second function. If in the integrand both the functions are easily integrable, then the first function is chosen in such a way that the derivative of the function is a simple function and the function thus obtained under the integral sign is easily integrable than the original function.
Usually, the following order is useful in deciding the first part. The function on the left is taken as the first part and the function on right is taken as second part.
Inverse, logarithmic, algebraic, trigonometric exponents
The above rule to decide between first and second part is known as I L A T E rule.

**Illustration - 9** $\int x\cos x\,dx =$

(A) $x\sin x + \cos x$ (B) $x\cos x + \sin x$ (C) $x(\sin x + \cos x)$ (D) *None of these*

**SOLUTION : (A)**

$$I = \int \underset{\text{part 1}}{x}\ \underset{\text{part 2}}{\cos x\,dx} = x\int \cos x\,dx - \int [\cos x\,dx]\,dx$$

$$\Rightarrow\quad I = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

***Illustrating the Concepts :***

*Study the following examples carefully .*

(i) $\int x \sec^2 x\, dx$ (ii) $\int \sin^{-1} x\, dx$ (iii) $\int \tan^{-1} x\, dx$ (iv) $\int x e^x\, dx$ (v) $\int \log x\, dx$

(vi) $\int x^2 \sin x\, dx$

(i) $\int x \sec^2 x\, dx = x\int \sec^2 x\, dx - \int \tan x\, dx$

$= x \tan x - \log|\sec x| + C$

(ii) $\int \sin^{-1} x\, dx$

$= \sin^{-1} x \int dx - \int \frac{x\, dx}{\sqrt{1-x^2}}$

$= x\sin^{-1} x - \frac{1}{2}\int \frac{2x}{\sqrt{1-x}} dx$

$= x \sin^{-1} x + \frac{1}{2}\int \frac{dt}{\sqrt{t}}$ where $1 - x^2 = t$

$= x \sin^{-1} x + \frac{1}{2} 2\sqrt{t} + C = x\sin^{-1} x + \sqrt{1-x^2} + C$

(iii) $\int \tan^{-1} x\, dx = \tan^{-1} x \int dx - \int \frac{x}{1+x^2} dx$

$x\tan^{-1} x - \frac{1}{2}\log|1+x^2| + C$

(iv) $\int x e^x\, dx = x\int e^x dx - \int e^x dx = xe^x - e^x + C$

(v) $\int \log x\, dx = \log x \int dx - \int x \frac{1}{x} dx = x\log x - x + C$

(vi) $\int x^2 \sin x\, dx = x^2 \int \sin x\, dx - \int (-\cos x)\, 2x\, dx$

$= -x^2 \cos x + 2\int x \cos x\, dx$

$= -x^2 \cos x + 2\left[x\int \cos x\, dx - \int \sin x\, dx\right]$

$= -x^2 \cos x + 2x \sin x + 2\cos x + C$

**Illustration - 10**

*If* $\int x \sin^{-1} x\, dx = \frac{2x^2-1}{4} f(x) + \frac{x}{4} g(x) + C$, *then :*

**(A)** $f(x) = \sin^{-1} x, g(x) = \frac{1}{\sqrt{1-x^2}}$

**(B)** $f(x) = \sin^{-1} x, g(x) = \sqrt{1-x^2}$

**(C)** $f(x) = \tan^{-1} x, g(x) = \frac{1}{\sqrt{1-x^2}}$

**(D)** $f(x) = \tan^{-1} x, g(x) = \sqrt{1-x^2}$

**SOLUTION : (B)**

$$\int \sin^{-1}x\, x\,dx = \sin^{-1}x\int x\,dx - \int \frac{x^2dx}{2\sqrt{1-x^2}} = \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\sqrt{1-x^2}\,dx - \frac{1}{2}\int\frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right] - \frac{1}{2}\sin^{-1}x + C$$

$$= \frac{2x^2-1}{4}\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} + C$$

## 3.3 Integrals of the Type $\int e^{ax}\sin bx\,dx, \int e^{ax}\cos bx\,dx$

Let $I = \int e^{ax}\sin bx\,dx$

Integrate by parts taking $e^{ax}$ as the first part, we get

$$I = \frac{-e^{ax}\cos bx}{b} - \int ae^{ax}\left(\frac{-\cos bx}{b}\right)dx$$

On integrating the second term by parts again, we get

$$I = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left[\frac{1}{b}e^{ax}\sin bx - \int\frac{a}{b}e^{ax}\sin bx\,dx\right]$$

$$\Rightarrow \quad I = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin x - \frac{a^2}{b^2}I \qquad \Rightarrow \qquad \left(1+\frac{a^2}{b^2}\right)I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx)$$

$$\Rightarrow \quad I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Similarly show that $\int e^{ax}\cos bx = \dfrac{e^{ax}}{a^2+b^2}\,(a\cos bx + b\sin bx)$

**Illustration - 11** $\int e^x \sin x\,dx =$

(A) $\dfrac{e^x}{2}(\sin x + \cos x)$  (B) $\dfrac{e^x}{2}(\sin x - \cos x)$

(C) $\dfrac{e^x}{2}(\cos x - \sin x)$  (D) $\dfrac{e^x}{2}(-\cos x - \sin x)$

**SOLUTION : (B)**

Let $I = \int e^x \sin x \, dx$

$\Rightarrow I = \int \sin x \; e^x \, dx = \sin x \int e^x \, dx - \int e^x [\cos x \, dx]$

$\Rightarrow I = e^x \sin x - \int \cos x \, e^x \, dx$

$I = e^x \sin x - \left[\cos x \int e^x \, dx - \int e^x \, (-\sin x \, dx)\right]$

$I = e^x (\sin x - \cos x) - \int e^x \sin x \, dx$

$I = e^x (\sin x - \cos x) - I$

$I + I = e^x (\sin x - \cos x)$

$I = e^x / 2 (\sin x - \cos x) + C$

**Illustration - 12** $\int \sec^3 x \, dx =$

**(A)** $\sec x \tan x + \log|\sec x + \tan x|$

**(B)** $\frac{1}{2}\left[\sec x \tan + \log|\sec x + \tan x|\right]$

**(C)** $\sec^2 x \tan x + \log|\sec x + \tan x|$

**(D)** $\frac{1}{2}\left[\sec^2 x \tan + \log|\sec x + \tan x|\right]$

**SOLUTION : (B)**

Let $I = \int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$

$= \sec x \int \sec^2 x \, dx - \int \tan x \, (\sec x \tan x) \, dx$

$\Rightarrow I = \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$

$= \sec x \; \tan x - I + \log|\sec x + \tan x|$

$\Rightarrow I = \sec x \; \tan x - I + \log|\sec x + \tan x|$

$\Rightarrow I = 1/2 \left[\sec x \tan x + \log|\sec x + \tan x|\right] + C$

## 3.4 Integrals of the Type $\int e^x [f(x) + f'(x)] dx$

Let $I = \int e^x [f(x) + f'(x)] \, dx \qquad \Rightarrow \qquad I = \int e^x \; f(x) \, dx + \int e^x f'(x) \, dx$

Integrate the first integral on the RHS by parts taking $e^x$ as the second function get

$$I = e^x f(x) - \int e^x f'(x) \, dx + \int e^x \; f'(x) \, dx \qquad \Rightarrow \qquad I = e^x f(x)$$

Thus to evaluate the integrals of the type $\int e^x \; g(x) \, dx$, we first try to express $g(x)$ as the sum of a function and its derivative i.e. $g(x) = f(x) \; f'(x)$ and then we use the result derived above.

***Illustrating the Concepts :***

*Evaluate the following :*

(i) $\int e^x \left[\frac{2+\sin 2x}{1+\cos 2x}\right] dx$ (ii) $\int \frac{xe^x}{(1+x)^2} dx$ (iii) $\int \frac{e^x (x^2+1)}{(x+1)^2} dx$ (iv) $\int \left[\log(\log x) + \frac{1}{\log x}\right] dx$

(i) $$I = \int e^x \left[\frac{2+\sin 2x}{1+\cos 2x}\right] dx$$

$$\Rightarrow \quad I = \int e^x \left[\frac{2}{1+\cos 2x} + \frac{\sin 2x}{1+\cos 2x}\right] dx$$

$$\Rightarrow \quad I = \int e^x \left[\frac{2}{2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x}\right] dx$$

$$= \int e^x [\sec^2 + \tan x]\, dx$$

$$\Rightarrow \quad I = \int e^x [\tan x + \sec^2 x]\, dx = e^x \tan x + C$$

(ii) $$I = \int \frac{x e^x}{(1+x)^2} dx = \int \frac{(1+x-1)}{(1+x)^2} e^x dx$$

$$\Rightarrow I = \int e^x \left[\frac{1}{1+x} - \frac{1}{(1+x)^2}\right] dx = e^x \left(\frac{1}{1+x}\right) + C$$

(iii) $$I = \int \frac{e^x (x^2+1)}{(x+1)^2} dx = \int e^x \left[\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}\right] dx$$

$$\Rightarrow \quad I = \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right] dx$$

We now say that

$$\frac{d}{dx}\left(\frac{x-1}{x+1}\right) = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$I = e^x \left[\frac{x-1}{x+1}\right] + C$$

(iv) $$I = \int \left[\log(\log x) + \frac{1}{\log x}\right] dx$$

Substitute

$$\log x = t \quad \Rightarrow \quad x = e^t \quad \text{and} \quad dx = c^t\, dt$$

$$\Rightarrow \quad \int \left(\log t + \frac{1}{t}\right) e^t\, dt = e^t \log t + C$$

$$\Rightarrow \quad e^{\log x} \log\log x + C = x \log\log x + C$$

## IN-CHAPTER EXERCISE - D

*Evaluate the following Integrals.*

(i) $\int x \log (x+1)\, dx$

(ii) $\int \frac{\log x}{x^2}\, dx$

(iii) $\int \frac{x}{1+\sin x} dx$

(iv) $\int \cos (\log x)\, dx$

(v) $\int \log \left(x+\sqrt{x^2+a^2}\right) dx$

(vi) $\int \frac{\log x}{(1+\log x)^2}\, dx$

(vii) $\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$

(viii) $\int e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$

(ix) $\int \sqrt{x}\, (\log x)^2\, dx$

(x) $\int x^2\, a^x\, dx$

(xi) $\int \log (\sqrt{1-x}\, \sqrt{1+x})\, dx$

(xii) $\int \cos x \log \left(\tan \frac{x}{2}\right) dx$

## INTEGRATION BY PARTIAL FRACTIONS Section - 4

The integrals of rational functions can be evaluated by splitting them into partial fractions. An expression containing a polynomial in numerator and another polynomial in denominator is called a rational function.

$\Rightarrow$ Rational Function $f(x) = \dfrac{\text{polynomial } N(x)}{\text{polynomial } D(x)}$

**Case-I :**

It the degree of numerator is less than the degree of denominator, we can split the rational functions into simpler fractions according to the factors of denominator.

For example :

$$\text{Let} \quad f(x) = \frac{x}{2x^2 - 5x - 3} \quad \Rightarrow \quad f(x) = \frac{x}{(x-1)(2x-3)} = \frac{-1}{x-1} + \frac{3}{2x-3}$$

$\dfrac{-1}{x-1}$ and $\dfrac{3}{2x-3}$ are known as partial fractions of $f(x)$.

The integral of $f(x)$ can now be evaluated as the sum of the integrals of its partial fractions.

$$\Rightarrow \quad \int f(x)\,dx = -\log|x-1| + 3/2 \log|2x-3| + C$$

**Case-II :**

If the degree of $N(x)$ is greater than or equal to the degree of $D(x)$, we divide $N(x)$ by $D(x)$ so that the rational functional $\dfrac{N(x)}{D(x)}$ is expressed in the form $Q(x) + \dfrac{R(x)}{D(x)}$ where degree of $R(x)$ less than the degree of $D(x)$.

Now as $Q(x)$ is a polynomial, it can be easily integrated and to integrate $\dfrac{R(x)}{D(x)}$ we make use of partial fractions as we have done in Case - I.

The resolution of $\dfrac{R(x)}{D(x)}$ into partial fractions depends upon the nature of the factors of denominator $D(x)$ as discussed below.

### 4.1 When denominator $g(x)$ is expressible as the product of non-repeating linear factors

Let $D(x) = (x - a_1)(x - a_2) \ldots\ldots\ldots (x - a_n)$.

Then we assume that

$$\frac{R(x)}{D(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a^2} + \ldots\ldots + \frac{A_n}{x - a_n}$$

where $A_1, A_2, \ldots\ldots A_n$ are constants and can be determined by equating the numerator on RHS to the numerator on LHS and the substitute $x = a_1, x = a_2, \ldots\ldots, x = a_n$ respectively.

**Illustration - 13** If $\int \frac{x^2\,dx}{(x-1)(2x+3)} = \frac{x}{2} + A\log|x-1| + B\log|2x+3| + C$, then :

**(A)** $A = \frac{1}{5}, B = \frac{9}{20}$ **(B)** $A = \frac{-1}{5}, B = \frac{9}{20}$

**(C)** $A = \frac{1}{5}, B = \frac{-9}{20}$ **(D)** $A = \frac{-1}{5}, B = \frac{-9}{20}$

**SOLUTION : (C)**

Let $I = \int \frac{x^2\,dx}{(x-1)(2x+3)}$

The degree of numerator is not less than the degree of denominator. Hence we devided $N$ by $D$.

$$\frac{x^2}{(x-1)(2x+3)} = \text{quotient} + \frac{\text{remainder}}{(x-1)(2x+3)}$$

$$= \frac{1}{2} + \frac{-\frac{1}{2}x + \frac{3}{2}}{(x-1)(2x+3)} = \frac{1}{2} + \frac{1}{2}\frac{3-x}{(x-1)(2x+3)}$$

We now split $\frac{3-x}{(x-1)(2x+3)}$ in two partial fractions.

Let $f(x)\ \frac{3-x}{(x-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{2x+3}$

where $A$ and $B$ are constants.

Equating the numerators on both sides :

$3 - x = A(2x + 3) + B(x - 1)$

Now there are two ways to calculate $A$ & $B$.

1. Comparing the coefficients of like terms.
2. Substituting the appropriate values of $x$.

Method 1 :

Comparing the coefficients of $x$ and $x^0$, we get :

$-1 = 2A + B$ and $3 = 3A - B$

On solving, we have $a = 2/5$ $B = -9/5$

Method 2 :

In $3 - x = A(2x + 3) + B(x - 1)$, put $x = 1, -3/2$

$x = 1 \Rightarrow 3 - 1 = 5A$

$\Rightarrow A = 2/5$

$x = -3/2 \Rightarrow 3 + 3/2 = B(-3/2 - 1)$

$\Rightarrow B = -9/5$

Hence finally we have :

$$f(x)\ \frac{\frac{2}{5}}{x-1} + \frac{-\frac{9}{5}}{2x+3} \Rightarrow I + \int\left[\frac{1}{2} + \frac{1}{2}f(x)\right]dx$$

$$\Rightarrow I = \frac{x}{2} + \frac{1}{2}\int\frac{\frac{2}{5}}{x-1}dx + \frac{1}{2}\int\frac{-\frac{9}{5}}{2x+3}dx$$

$$\Rightarrow I = \frac{x}{2} + \frac{1}{5}\log|x-1| - \frac{9}{20}\log|2x+3| + C$$

***Note :*** If the denominator contains only linear factors, the constants $A$ and $B$ can be calculated by the following method also.

$$F(x) = \frac{3-x}{(x-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{2x+3}$$

$A$ = Value obtained by substituting $x = 1$ in $f(x)$ after removing $(x - 1)$ from denominator

$$\Rightarrow A = \left.\frac{3-x}{2x+3}\right]_{x=1} = \frac{3-1}{2+3} = \frac{2}{5}$$

B = Value obtained by substituting $x=-\frac{3}{2}$ in $f(x)$ after removing $(2x+3)$ from denomainator.

$$\Rightarrow B = \left.\frac{3-x}{x-1}\right]_{x=-\frac{3}{2}} = \frac{3+\frac{3}{2}}{\frac{-3}{2}-1} = -\frac{9}{5}$$

**Illustration - 14**

*If* $\int \frac{(x-1)\,dx}{(2x+1)(x-2)(x-3)} = A\log|2x+1| + B\log|x-2| + C\log|x-3| + D$, *then* :

(A) $A=\frac{-3}{35}, B=\frac{-1}{5}, C=\frac{2}{7}$

(B) $A=\frac{3}{35}, B=\frac{1}{5}, C=\frac{2}{7}$

(C) $A=\frac{-3}{35}, B=\frac{-1}{5}, C=\frac{-2}{7}$

(D) $A=\frac{3}{35}, B=\frac{1}{5}, C=\frac{-2}{7}$

**SOLUTION : (A)**

Let $f(x) = \int \frac{x-1}{(2x+1)(x-2)(x-3)}$

$$= \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$\Rightarrow A = \left.\frac{x-1}{(x-2)(x-3)}\right]_{x=-\frac{1}{2}} = -\frac{6}{35}$$

$$\Rightarrow B = \left.\frac{x-1}{(2x+1)(x+3)}\right]_{x=2} = -\frac{1}{5}$$

$$\Rightarrow C = \left.\frac{x-1}{(2x+1)(x-2)}\right]_{x=3} = \frac{2}{7}$$

$$\Rightarrow \int f(x)dx = \frac{-6}{35}\int \frac{dx}{2x+1} - \frac{1}{5}\int \frac{dx}{x-2} + \frac{2}{7}\int \frac{dx}{x-3}$$

$$= -\frac{3}{35}\log|2x+1| - \frac{1}{5}\log|x-2| + \frac{2}{7}\log|x-3| + C$$

## 4.2 When denominator $g(x)$ is expressible as the product of linear factors such that some of them are repeating

Let $D(x) = (x-a)^k (x-a_1)(x-a_2)\,.........\,(x-a_n)$.

Then we assume that

$$\frac{R(x)}{D(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} \ldots\ldots + \frac{A_k}{(x-a)^k} + \frac{B_1}{x-a_1} + \frac{B_2}{x-a_2} + \ldots\ldots\ldots \frac{B_n}{x-a_n}$$

i.e. corresponding to the non-repeating factors we assume as in Section 4.1 and for each repeating factors of the type $(x-a)^k$, we assume partial fractions of the type :

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}\cdots\cdots\cdot+\frac{A_k}{(x-a)^k}$$ where $A_1, A_2, \ldots\ldots A_n$ are constants.

Now to determine constants we equate numerators on both sides. Then from the resulting identity we can find all constants by substituting the various values of $x$ as we have done in Section 4.1.

**Illustration - 15** $\displaystyle\int \frac{(\cos\theta+1)\sin\theta}{(\cos\theta-1)^2(\cos\theta-3)}\,d\theta=\log\left|\frac{\cos\theta-1}{\cos\theta-3}\right|+f(\theta)+C$, *then* $f(\theta)=$

(A) $\dfrac{1}{\sin\theta-1}$ (B) $-\dfrac{1}{\sin\theta-1}$ (C) $\dfrac{1}{\cos\theta-1}$ (D) $-\dfrac{1}{\cos\theta-1}$

**SOLUTION : (D)**

Let $\cos\theta = x \quad\Rightarrow\quad -\sin\theta\, d\theta = dx$

$$\Rightarrow\; I=-\int\frac{x+1}{(x-1)^2(x-3)}\,dx$$

Let $f(x)=$

$$\int\frac{x+1}{(x-1)^2(x-3)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-3}$$

Equating numerator on both sides,

$\Rightarrow\; x+1$

$= A(x-1)(x-3)+B(x-3)+C(x-1)^2$

By taking $x = 1$, we get $B = -1$.

By taking $x = 3$, we get $C = 1$.

Comparing the coefficient of $x^2$, we get,

$0 = A + C \;\Rightarrow\; 0 = A + 1 \;\Rightarrow\; A = -1$

$\Rightarrow\; I=-\int f(x)\,dx$

$$=-\left\{\int\frac{-1}{x-1}dx+\int\frac{-1}{(x-1)^2}dx+\int\frac{1}{x-3}dx\right\}$$

$$\Rightarrow\; I=\log|x-1|-\frac{1}{x-1}-\log|x-3|+C$$

$$\Rightarrow\; I=\log\left|\frac{x-1}{x-3}\right|-\frac{1}{x-1}+C$$

$$\Rightarrow\; I=\log\left|\frac{\cos\theta-1}{\cos\theta-3}\right|-\frac{1}{\cos\theta-1}+C$$

## 4.3 When some of the factors of denominator D (x) are quadratic (which can not be factorised ruther) but not repeating.

Corresponding to each quadratic factor of the type $ax^2 + bx + c$, we assume partial fractions of the type

$\dfrac{Ax+B}{ax^2+bx+c}$, where $A$ and $B$ are constants to be determined comparing coefficients of similar power of $x$ in the numerator of both sides. (The constants can also be determined using methods which we have used in Sections 4.1 and 4.2).

**Illustration - 16**

$$If \int \frac{d}{x^3+1} = A\log\left|\frac{x+1}{\sqrt{x^2-x+1}}\right| + B\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C, \ then:$$

(A) $A=\frac{1}{\sqrt{3}}, B=\frac{1}{\sqrt{3}}$ (B) $A=\frac{1}{3}, B=\frac{1}{\sqrt{3}}$

(C) $A=\frac{1}{\sqrt{3}}, B=\frac{1}{3}$ (D) $A=\frac{1}{3}, B=\frac{1}{3}$

**SOLUTION : (B)**

Let $f(x) = \dfrac{1}{x^3+1} = \dfrac{1}{(x+1)(x^2-x+1)}$

$\Rightarrow$ $f(x) = \dfrac{1}{(x+1)(x^2-x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}$

$\Rightarrow$ $1 = a(x^2 - x + 1) + (Bx + C)(x + 1)$

Comparing the coefficients of $x^2, x, x^0$.

$0 = A + B$, $0 = -A + B + C$ $1 = A + C$

$\Rightarrow$ $A = 1/3$, $C = 2/3$, $B = -1/3$

$\Rightarrow$ $f(x) = \dfrac{\frac{1}{3}}{x+1} + \dfrac{-\frac{x}{3}+\frac{2}{3}}{x^2-x+1}$

Let $I_1 = \dfrac{1}{3}\displaystyle\int \dfrac{dx}{x+1} = \dfrac{1}{3}\log|x+1| + C_1$

Let $I_2 = \displaystyle\int \dfrac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}\,dx = \dfrac{1}{3}\int \dfrac{2-x}{x^2-x+1}\,dx$

Express the numerator in terms of derivative of denominator.

$\Rightarrow$ $I_2 = -\dfrac{1}{6}\displaystyle\int \dfrac{2x-4}{x^2-x+1}\,dx$

$\Rightarrow$ $I_2 = -\dfrac{1}{6}\displaystyle\int \dfrac{2x-1}{x^2-x+1}\,dx + \dfrac{1}{2}\int \dfrac{dx}{x^2-x+1}$

$$\Rightarrow \quad I_2 = -\frac{1}{6}\log|x^2 - x + 1| + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$\Rightarrow \quad I_2 = -\frac{1}{6}\log|x^2 - x + 1| + \frac{2}{2\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C_2$$

$$\Rightarrow \quad I_2 = -\frac{1}{6}\log|x^2 - x + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C_2$$

$$\Rightarrow \quad \int \frac{dx}{x^3+1} = \int f(x)\,dx = I_1 + I_2 = \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2 - x + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

$$= \frac{1}{3}\log\left|\frac{x+1}{\sqrt{x^2 - x + 1}}\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

## 4.4 When some of the factors of denominator D (x) are quadratic (which can into be factorised further) and repeating

For every quadratic factor of the type $(ax^2 + bx + c)$, we assuming partial fractions of the type :

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \ldots\ldots + \frac{A_nx + B_n}{(ax^2 + bx + C)^n}$$

The constants are determined using method which we have already discussed Sections 4.1 and 4.2.

## 4.5 When integrand contains only even powers of *x*

The integral in which the integral contains only the even powers of $x$ can be evaluated through the following steps :

Step – I : Consider integrand and put $x^2 = t$ in it.

Step – II : Make partial fractions of the resulting rational expression in $t$.

Step – III : Put $t = x^2$ again in the partial fraction and then integrate both sides.

**Illustration - 17**

If $\int \frac{x^2\,dx}{x^4-1} = A\log\left|\frac{x-1}{x+1}\right| + B\tan^{-1}x + C$, *then* :

(A) $A=\frac{1}{2},\ B=\frac{1}{2}$ (B) $A=\frac{1}{4},\ B=\frac{1}{2}$

(C) $A=\frac{1}{2},\ B=\frac{1}{4}$ (D) $A=\frac{1}{4},\ B=\frac{1}{4}$

**SOLUTION : (B)**

$$\int \frac{x^2\,dx}{x^4-1} = \int \frac{x^2\,dx}{(x^2-1)(x^2+1)}$$

As the function contains terms of $x^2$ only, substitute $x^2 = t$ and then make partial fractions.

$$\frac{t}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1}$$

$\Rightarrow \quad t = A(t+1) + B(t-1)$

Put $t = \pm 1$ to get $A = 1/2,\ B = 1/2$.

$$\Rightarrow \quad \frac{t}{(t-1)(t+1)} = \frac{\frac{1}{2}}{t-1} + \frac{\frac{1}{2}}{t+1}$$

Covert $t = x^2$ again before integrating.

$$\Rightarrow \quad I = \int \frac{x^2\,dx}{(x^2-1)(x^2-1)} = \int \frac{1/2}{x^2-1}\,dx + \int \frac{1/2}{x^2+1}\,dx$$

$$= \frac{1}{2}\,\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\tan^{-1}x + C$$

## IN-CHAPTER EXERCISE - E

*Evaluate the following Integrals.*

(i) $\int \frac{dx}{x^3-1}$ (*ii*) $\int \frac{3x+5}{x^3-x^2-x+1}\,dx$ (iii) $\int \frac{x}{(1+x)(1+x^2)}\,dx$

(iv) $\int \frac{x-1}{(x+1)(x^2+1)}\,dx$ (v) $\int \frac{dx}{(x-1)(x^2+1)}$ (vi) $\int \frac{x^4}{(x-1)(x^2+1)}\,dx$

(vii) $\int \frac{x}{(x^2-a^2)(x^2-b^2)}\,dx$ (viii) $\int \frac{x^2+1}{(x^2+2)(2x^2+1)}\,dx$ (ix) $\int \frac{1+x+x^2}{x^3(1+x)}\,dx$

## INTEGRATION OF IRRATIONAL ALGEBRAIC FUNCTIONS  Section - 5

### 5.1 Integrals of the type : $\int \frac{dx}{\text{linear}\sqrt{\text{linear}}}$

To evaluate these type of integrals we substitute $\sqrt{\text{linear}} = t$. This substitution will reduce the integral to one of the known forms.

**Illustration - 18** If $\int \frac{dx}{(x+1)\sqrt{x+2}} = \log\left| f(x)\right| + C$, then $f(x) =$

(A) $\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$  (B) $\frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}$

(C) $\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}$  (D) $\frac{\sqrt{x+2}+1}{\sqrt{x+2}-1}$

**SOLUTION : (C)**

Let $I = \int \frac{ax}{(x+1)\sqrt{x+2}}$

Substitute : $x + 2 = t^2$ $\Rightarrow$ $dx = 2t\, dt$

$$\Rightarrow \int \frac{dx}{x+1\left(\sqrt{x+2}\right)} = \int \frac{2t\, dt}{(t^2-1)\sqrt{t^2}} = 2\int \frac{dt}{t^2-1}$$

$$= \frac{2}{2}\log\left|\frac{t-1}{t+1}\right| + C = \log\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + C$$

### 5.2 Integral of the type : $\int \frac{dx}{\text{quadratic}\sqrt{\text{linear}}}$

To evaluate these type of integrals we substitute $\sqrt{\text{linear}} = t$. This substitution will reduce the integral to one of the known forms.

**Illustration - 19** If $\int \frac{x}{(x^2-3x+2)\sqrt{x-1}}dx = 2\log\left|\frac{\sqrt{x-1}-1}{\sqrt{x-1}+1}\right| + f(x) + C$, then $f(x) =$

(A) $\frac{1}{\sqrt{x-1}}$  (B) $\frac{1}{\sqrt{x+1}}$  (C) $\frac{2}{\sqrt{x-1}}$  (D) $\frac{2}{\sqrt{x+1}}$

**SOLUTION : (C)**

Let $x - 1 = t^2 \quad \Rightarrow \quad dx = 2t\,dt$

$$\Rightarrow I = \int \frac{(t^2+1)}{(t^2+1)^2 - 3(t^2+1) + 2} \frac{2t\,dt}{\sqrt{t^2}}$$

$$= 2\int \frac{(t^2+1)\,dt}{t^4+t^2}$$

$$\Rightarrow I = 2\int \frac{t^2+1}{t^2\left(t^2-1\right)}dt = \int\left(\frac{2}{t^2-1} - \frac{1}{t^2}\right)dt$$

$$= 4\int \frac{dt}{t^2-1} - 2\int \frac{dt}{t^2}$$

$$\Rightarrow I = \frac{4}{2}\log\left|\frac{t-1}{t+1}\right| + \frac{2}{t} + C$$

$$\Rightarrow I = 2\log\left|\frac{\sqrt{x-1}-1}{\sqrt{x-1}+1}\right| + \frac{2}{\sqrt{x+1}} + C$$

## 5.3 Integral of the type : $\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$

To evaluate these type of integrals we substitute $x = \frac{1}{t}$. This substitution will reduce the integral to one of the known forms.

**Illustration - 20**

If $\int \frac{dx}{(x^2+1)\sqrt{x^2+2}} = -\tan^{-1}(f(x)) + C$, then $f(x) =$

(A) $\sqrt{x^2+1}$ (B) $\sqrt{1+\frac{1}{x^2}}$ (C) $\sqrt{x^2+2}$ (D) $\sqrt{1+\frac{2}{x^2}}$

**SOLUTION : (D)**

Let $I = \int \frac{dx}{(x^2+1)\sqrt{x^2+2}}$

Substitute : $x = \frac{1}{t} \quad \Rightarrow \quad dx = \frac{1}{t^2}dt$

$$\Rightarrow I = \int \frac{-\frac{1}{t^2}dt}{\left(\frac{1}{t^2}+1\right)\sqrt{\frac{1}{t^2}+2}} = \int \frac{-tdt}{\left(1+t^2\right)\sqrt{1+2t^2}}$$

Let $1 + 2t^2 = z^2 \quad \Rightarrow \quad 4\,t\,dt = 2\,z\,dz$

$\Rightarrow$

$$I = \frac{-1}{2}\int \frac{zdz}{\left(1+\frac{z^2-1}{2}\right)\sqrt{z^2}} = \int \frac{-dz}{z^2+1} = -\tan^{-1} z + C$$

$$\Rightarrow I = -\tan^{-1}\sqrt{1+2t^2} + C = -\tan^{-1}\sqrt{1+\frac{2}{x^2}} + C$$

## 5.4 Integral of the type : $\int \frac{dx}{\text{linear}\sqrt{\text{quadratic}}}$

To evaluate these type of integrals we substitute Linear = $\frac{1}{t}$. The substitution will reduce the integral to one of the know forms.

**Illustration - 21** If $\int \frac{dx}{(x+2)\sqrt{x^2+6x+7}} = \sin^{-1}(f(x)) + C$, then $f(x) =$

(A) $\frac{1}{\sqrt{2}(x+1)}$ (B) $\frac{1}{\sqrt{2}(x+2)}$ (C) $\frac{x+1}{\sqrt{2}(x+2)}$ (D) $\frac{x+1}{\sqrt{2}(x+1)}$

**SOLUTION : (D)**

Let $I = \int \frac{dx}{(x+2)\sqrt{x^2+6x+7}}$

Substitute :

$x + 2 = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}$

$\Rightarrow x^2 + 6x + 7$

$= \left(\frac{1}{t}-2\right)^2 + 6\left(\frac{1}{t}-2\right) + 7 = \frac{1+2t-t^2}{t^2}$

$\Rightarrow I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1+2t+t^2}{t^2}}} - -\int \frac{dt}{\sqrt{1+2t-t^2}}$

$\Rightarrow I = \sin^{-1}\left[\frac{x+1}{(x+2)\sqrt{2}}\right] + C$

## 5.5 Integral of the type : $\int f(x,(ax+b)^{m_1/n_1},(ax+b)^{m_2/n_2},.......)dx$

where $f$ is a irrational function and $m_1$, $n_1$, $m_2$, $n_2$ are integers.
To evaluate this type of integral we transform it into an integral of rational function by substituting $ax + b = t^s$, where $s$ is the least common multiple of the numbers $n_1$, $n_2$, . . . . . .

**Illustration - 22** If $\int \frac{dx}{\sqrt[3]{x+1}+\sqrt{x+1}} = k\left(\frac{(1+x)^{1/2}}{3} - \frac{(1+x)^{1/3}}{2} + (1+x)^{1/6} - \log\left|(1+x)^{1/6}+1\right|\right) + C$, then $k$ :

(A) 1 (B) 2 (C) 3 (D) 6

**SOLUTION :**

Let $I = \int \frac{dx}{\sqrt[3]{x+1} + \sqrt{x+1}}$

$\Rightarrow \quad I = \int \frac{dx}{(x+1)^{1/3} + (x+1)^{1/2}}$

The least common multiple of 2 and 3 is 6.

So substitute $x + 1 = t^6 \quad \Rightarrow \quad dx = 6t^5\, dt$

$\Rightarrow \quad I = \int \frac{6t^5\, dt}{t^2 + t^2} = 6\int \frac{t^3 dt}{1+t}$

$\Rightarrow \quad I = 6\int \left( t^2 - t + 1 - \frac{1}{1+t} \right) dt$

$\Rightarrow \quad I = 6\left( \frac{t^2}{3} - \frac{t^2}{2} + t - \log(t+1) \right) + C$

On Substituting $t = (1+x)^{1/6}$, we get

$$I = 6\left( \frac{(1+x)^{1/2}}{3} - \frac{(1+x)^{1/3}}{2} + (1+x)^{1/6} - \log \right.$$

$$\left. - \log\left| (x+1)^{1/6} + 1 \right| \right) + C$$

## 5.6 Integral of the type : $\int x^m (a + bx^n)^p\, dx,$

where $m$, $n$ and $p$ are rational numbers.

To evaluate this type of integral observe the following steps :

(i) If $p$ is an integer, the integral reduces to the integral of a rational function by means of the substitution $x = t^s$, where $s$ is least common multiple of the denominator of the fractions $m$ and $n$.

(ii) If $(m + 1)/n$ is an integer, the integral can be rationalised by the substitution $a + bx^n = t^s$, where $s$ is the denominator of the fraction $p$.

(iii) If $(m + 1)/n + p$ is an integer, substitute $ax^{-n} + b = t^s$, where $s$ is the denominator of the fraction $p$.

**Illustration - 23**

$$If \int x^{13/2}\left(1 + x^{5/2}\right)^{1/2} dx = k\left( \frac{(1+x^{5/2})^{7/2}}{7} + \frac{(1+x^{5/2})^{3/2}}{3} + \frac{2(1+x^{5/2})^{5/2}}{5} \right) + C,\ then\ k =$$

(A) $2/5$ (B) $4/5$ (C) $5/2$ (D) $5/4$

**SOLUTION :**

Let $I = \int x^{13/2}\left(1 + x^{5/2}\right)^{1/2} dx$

Comparing with integral of type 5.6, we see that $p = 1/2$ which is not an integer.

So this integral does not belong to type 5.6 (i).

Check the sign of $(m + 1)/n$

$$\frac{m+1}{n} = \frac{\frac{13}{2}+1}{\frac{5}{2}} = \frac{15}{5} = 3$$

$\Rightarrow$ $(m + 1)/n$ is an integer. So this integral belongs to type 5.6 (ii).

To solve this integral, substitute $1 + x^{5/2} = t^2$

$\Rightarrow \quad 5/2x^{3/2}\, dx = 2t\, dt$

$\Rightarrow \quad I = \frac{2}{5}\int (t^2 - 1)^2\, (t^2)^{1/2}\, 2t\, dt$

$\Rightarrow \quad I = \frac{4}{5}\int t^2 (t^2 - 1)^2 dt$

$$\Rightarrow \quad I = \frac{4}{5}\int t^6 + t^2 - 2t^4 \, dt$$

$$\Rightarrow \quad I = \frac{4}{5}\left(\frac{t^7}{7} + \frac{t^3}{3} - 2\frac{t^5}{5}\right) + C$$

On substituting $t = (1 + x^{5/2})^{1/2}$, we get

$$I = \frac{4}{5}\left(\frac{(1+x^{5/2})^{7/2}}{7} + \frac{(1+x^{5/2})^{3/2}}{3} + \frac{2\,(1+x^{5/2})^{5/2}}{5}\right) + C$$

## 5.7 Integration of irrational algebraic functions by trigonometric substitutions

Those Irrational functions which in involve either of $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$ . . . . . . . terms and no other radical, can be integrated by a suitable trigonometric substitution. The trigonometric substitutions are similar to those given in Section 2.2.

***Illustrating the Concepts :***

*Evaluate :* (i) $\dfrac{dx}{(2ax + x^2)^{3/2}}$ (ii) $\displaystyle\int \frac{\sqrt{x^2+1}}{x^4}\,dx$.

(i) *Let :* $I = \displaystyle\int \frac{dx}{(2ax + x^2)^{3/2}}$

$$\Rightarrow I = \int \frac{dx}{\left[(x-a)^2 - a^2\right]^{3/2}}$$

Put $x + a = a \sec\theta \quad \Rightarrow \quad dx = a \sec\theta \tan\theta \, d\theta$

On substituting in $I$, we get

$$I = \int \frac{a\sec\theta\tan\theta d\theta}{(a^2\sec^2\theta - a^2)^{3/2}} = \int \frac{a\sec\theta\tan\theta}{a^3\tan^3\theta}$$

$$\Rightarrow I = \frac{1}{a^2}\int \sec\theta\cot^2\theta d\theta = \frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$\Rightarrow I = \frac{1}{a^2}\int \frac{d(\sin\theta)}{\sin^2\theta}\,d\theta = -\frac{1}{a^2\sin\theta} + C$$

$$\Rightarrow I = -\frac{1}{a^2}\frac{x+a}{\sqrt{x^2+2ax}} + C$$

(ii) Let $I = \displaystyle\int \frac{\sqrt{x^2+1}}{x^4}\,dx$

Put $x = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta \, d\theta$

On substituting $x$ and $dx$ in I, we gets

$$I = \int \frac{\sqrt{\tan^2\theta + 1}}{\tan^4\theta}\sec^2\theta\,d\theta = \int \frac{\sec^3\theta\,d\theta}{\tan^4\theta}$$

$$\Rightarrow \quad I = \int \frac{\cos\theta}{\sin^4\theta}\,d\theta = \int \frac{d\,(\sin\theta)\,d\theta}{\sin^4\theta}$$

$$\Rightarrow \quad I = -\frac{1}{3\sin^3\theta} + C$$

On substituting value of $\sin\theta$ in terms of $x$, we get

$$I = -\frac{1}{3}\frac{(1+x^2)^{3/2}}{x^3} + C$$

## IN-CHAPTER EXERCISE - F

*Evaluate the following Integrals.*

(i) $\int \frac{x}{(x-3)\sqrt{x+1}}\,dx$

(ii) $\int \frac{2-3x}{x\sqrt{1+x}}\,dx$

(iii) $\int \frac{x^2+2}{(x^2+2x+2)\sqrt{x-1}}\,dx$

(iv) $\int \frac{dx}{(x+1)^{3/2}(x-1)^{1/2}}$

(v) $\int \frac{x^2+2x+3}{(x+1)\sqrt{x^2+1}}\,dx$

(vi) $\int \frac{dx}{(1+x)^2\sqrt{2+x^2}}$

(vii) $\int \frac{x}{(x^2+4)\sqrt{x^2+9}}\,dx$

(viii) $\int \frac{\sqrt{1+x^2}}{x}\,dx$

(ix) $\int \frac{x^2}{(4+x^2)^{3/2}}\,dx$

(x) $\int x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$

(xi) $\int \frac{1+x^{1/2}}{1+x^{1/3}}\,dx$

(xii) $\int \frac{\sqrt{x}}{(1+x)(2+x)(3+x)}\,dx$

## REDUCTION FORMULA — Section - 6

### 6.1 Introduction

Some functions occur whose integrals are not immediately reducible to one or other of the standard form and whose integrals are not directly obtainable In such cases we find out a formula (called reduction formula) which connects one integral with another in which the intergrand is of the same type but is of lower degree or order and hence easier to integrate.

The following examples will illustrate you the concept of deriving the reduction formulas for various functions :

**Illustration - 24**

If $I_n = \int \sin^n x\,dx$, then $I_n = \dfrac{-\cos x \sin^{n-1} x}{n} + f(n) I_{n-2}$, for $f(n) =$

(A) $\frac{n-1}{n}$ (B) $\frac{n}{n-1}$ (C) $\frac{n-2}{n}$ (D) $\frac{n}{n-2}$

**SOLUTION : (A)**

Let $I_n = \int \sin^n x\,dx = \int \sin^{n-1} x \cdot \sin x\,dx$

Apply by parts taking $\sin^{n-1} x$ as first part and $\sin x$ as second part.

$$\Rightarrow \quad I_n = \sin^{n-1} x \cdot (-\cos x) + \int (n-1)\sin^{n-2} x \cos^2 x\, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

$$\Rightarrow \quad I_n = -\cos x \sin^{n-1} x + (n-1)\, I_{n-2} - (n-1)\, I_n$$

$$\Rightarrow \quad nI_n = -\cos x \sin^{n-1} x + (n-1)\, I_{n-2}; \qquad I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$$

## Illustration - 25

*If* $I_n = \int \tan^n x\, dx$, *then* $(n-1)(I_n + I_{n-2}) =$

(A) $\tan^{n-2} x$ (B) $\tan^{n-1} x$ (C) $\tan^n x$ (D) $\tan^{n+1} x$

**SOLUTION : (B)**

Here $I_n = \int \tan^n x\, dx = \int \tan^{n-2} x \tan^2 x\, dx$

$$= \int \tan^{n-2} x\,(\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - I_{n-2}$$

$$\Rightarrow \quad I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}$$

Hence $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$.

## Illustration - 26

*If* $\int \sec^n x\, dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + f(n)\int \sec^{n-2} x\, dx$, *then* $f(n) =$

(A) $\dfrac{n}{n-1}$ (B) $\dfrac{n-1}{n}$ (C) $\dfrac{n-2}{n-1}$ (D) $\dfrac{n-1}{n-2}$

**SOLUTION : (C)**

Let $I_n = \int \sec^n x \quad \Rightarrow \quad I_n = \int \sec^{n-2} x \sec^2 x\, dx$

Apply by parts taking $\sec^{n-2} x$ as the first part and $\sec^2 x$ as the second part.

$$\Rightarrow \quad I_n = \sec^{n-2} x \int \sec^2 x\, dx - \int \left[ \frac{d}{dx}\left(\sec^{n-2} x\right) \int \sec^2 x\, dx \right] dx$$

$$\Rightarrow \quad I_n = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan x \tan x\, dx$$

$$\Rightarrow \quad I_n = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1)\, dx$$

$$\Rightarrow \quad I_n = \sec^{n-2} x \tan x - (n-2)\int \sec^n x\, dx + (n-2)\int \sec^{n-2} x\, dx$$

$$\Rightarrow \quad I_n + (n-2)\, I_n = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\, dx.$$

$$\Rightarrow \quad (n-1)\, I_n = \sec^{n-2} x \tan x + (n-2)\, I_{n-2}$$

Hence $\displaystyle\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx.$

## Illustration - 27

*If* $I_n = \int e^{ax} \cos^n x\, dx$, *then* $I_n = e^{ax}\left(\dfrac{f(x)}{a^2+n^2}\right)\cos^{n-1} x + \dfrac{n(n-1)}{a^2+n^2} I_{n-2}$ *for* $f(x) =$

**(A)** $a\cos x + n\sin x$ **(B)** $a\sin x + n\cos x$

**(C)** $a^2\cos x + n^2\sin x$ **(D)** $a^2\sin x + n^2\cos x$

**SOLUTION : (A)**

Let $I_n = \int e^{ax} \cos^n x\, dx$

Apply by part taking $\cos^n x$ as the first part and $e^{ax}$ as the second part.

$$\Rightarrow \quad I_n = \cos^n x \int e^{ax} dx - \int\left[\frac{d}{dx}\left(\cos^n x\right)\int e^{ax}\, dx\right] dx$$

$$\Rightarrow \quad I_n = \frac{e^{ax}}{a}\cos^n x - \int n\cos^{n-1} x\,(-\sin x)\,\frac{e^{ax}}{a}\, dx$$

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax}\cos^n x + \frac{n}{a}\int\left(\cos^{n-1} x \sin x\right) e^{ax} dx$$

Apply by parts again taking $\cos^{n-1} x \sin x$ as first part and $e^{ax}$ as second part.

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax}\cos^n x + \frac{n}{a}\left(\cos^{n-1} x \sin x\right)\int e^{ax} - \frac{n}{a}\int\left[\frac{d}{dx}\left(\cos^{n-1} x \sin x\right)\int e^{ax} dx\right] dx$$

$$\Rightarrow I_n = \frac{1}{a} e^{ax}\cos^n x + \frac{n}{a}\cos^{n-1} x \sin x\,\frac{e^{ax}}{a} - \frac{n}{a}\int\left[-(n-1)\cos^{n-2} x \sin^2 x + \cos^{n-1} x \cos x\right]\frac{e^{ax}}{a}\, dx$$

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax}\cos^n x + \frac{n}{a^2} e^{ax}\cos^{n-1} x \sin x + \frac{n(n-1)}{a^2}\int e^{ax}\cos^{n-2} x\,(1-\cos^2 x)\, dx - \frac{n}{a^2}\int e^{ax}\cos^n x\, dx$$

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax}\cos^n x + \frac{n}{a^2} e^{ax}\cos^{n-1} x \sin x + \frac{n(n-1)}{a^2} I_{n-2} - \frac{n^2}{a^2} I_n$$

$$\Rightarrow \quad I_n = \left(1+\frac{n^2}{a^2}\right) I_n = \frac{1}{a^2} e^{ax}(a\cos x + n\sin x)\cos^{n-1} x + \frac{n(n-1)}{a^2} I_{n-2}$$

Hence $\displaystyle\int e^{ax}\cos^n x\,dx = e^{ax}\left(\frac{a\cos x + n\sin x}{a^2+n^2}\right)\cos^{n-1} x - \frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2} x\,dx$

This is the required reduction formula.

## Illustration - 28

If $\displaystyle I_{m,n} = \int \cos^m x \sin nx\,dx = f(m,n) I_{m-1,n-1} - \frac{\cos^m x\cos x}{m+n}$, then $f(m,n) =$

(A) $\dfrac{m}{m+n}$ (B) $\dfrac{n}{m+n}$ (C) $\dfrac{m-1}{m+n}$ (D) $\dfrac{n-1}{m+n}$

**SOLUTION : (A)**

Let $\displaystyle I_{m,n} = \int \cos^m x \sin nx\,dx$

Apply by parts taking $\cos^m x$ as the first part and $\sin nx$ as the second part.

$$\Rightarrow \quad I_{m,n} = \cos^m x\left(-\frac{\cos nx}{n}\right) - \int m\cos^{m-1} x(-\sin x)\left(-\frac{\cos nx}{n}\right)dx$$

$$\Rightarrow \quad I_{m,n} = -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x(\sin x\cos nx)\,dx$$

Now $\sin(n-1)x = \sin nx\cos x - \cos nx\sin x$

$\Rightarrow \quad \cos nx\sin x = \sin nx\cos x - \sin(n-1)x.$

$$\Rightarrow \quad I_{m,n} = -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x[\sin nx\cos x - \sin(n-1)x]\,dx$$

$$\Rightarrow \quad I_{m,n} = -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}\int \cos^m x\sin nx\,dx + \frac{m}{n}\int \cos^{m-1} x\sin(n-1)x\,dx$$

$$\Rightarrow \quad \left[1+\frac{m}{n}\right] I_{m,n} = -\frac{\cos^m x\cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \quad I_{m,n} = \frac{m}{m+n} I_{m-1,n-1} - \frac{\cos^m x\cos nx}{m+n}$$

## IN-CHAPTER EXERCISE - G

1. *Find the reduction formulas for the following functions.*

(i) $\int \cos^n x \, dx$ (ii) $\int \cot^n x \, dx$ (iii) $\int \operatorname{cosec}^n x \, dx$

2. If $I_{m,n} = \int \cos^m x \cos nx \, dx$, then show that : $I_{m,n} = \dfrac{\cos^m x \sin nx}{m+n} + \dfrac{m}{m+n} I_{m-1,\, n-1}$

3. *Find the formula for* $\int e^{ax} \sin^n x \, dx$.

### Evaluate the following :

4. $\int [\sqrt{\tan x} + \sqrt{\cot x}]\, dx$

5. $\int \dfrac{1}{1 + 3\sin^2 x} dx$

6. $\int \dfrac{d\theta}{(\sin\theta - 2\cos\theta)(2\sin\theta + \cos\theta)}$

7. $\int \dfrac{1}{p + q\tan x} dx$

8. $\int \dfrac{1}{\sin^2 x + \sin 2x} dx$

9. $\int \dfrac{2 - \sin x}{2\cos x + 3} dx$

10. $\int \dfrac{1}{1 - \cot x} dx$

11. $\int \dfrac{e^x}{e^x + e^{-x}} dx$

12. $\int \sqrt{\dfrac{a+x}{a-x}}\, dx$

13. $\int \dfrac{1}{\sin(x-a)\sin(x-b)} dx$

14. $\int \dfrac{3x+5}{x^3 - x^2 - x + 1} dx$

15. $\int \dfrac{x}{(1+x)(1+x^2)} dx$

16. $\int \dfrac{1}{\sin x(5 + 4\cos x)} dx$

17. $\int x\sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}\, dx$

18. $\int \dfrac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$

19. $\int [1 + 2\tan x(\tan x + \sec x)]^{1/2} dx$

20. $\int \log(1 + x^2)\, dx$

21. $\int \sin x \log(\sec x + \tan x)\, dx$

22. $\int \dfrac{\cos x}{\cos(x-a)} dx$

23. $\int \dfrac{1}{\cos(x-a)\cos(x-b)} dx$

24. $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$

25. $\int \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$

26. $\int \frac{x-\sin x}{1-\cos x}\,dx$

27. $\int \frac{\log x}{x^3}\,dx$

28. $\int \frac{1}{\sqrt{(x+a)}+\sqrt{x+b}}\,dx$

29. $\int \frac{1}{\log x^x\,[(\log x)^2-3\log x-10]}\,dx$

30. $\int \frac{x}{\sqrt{x+a}+\sqrt{x+b}}\,dx$

31. $\int \frac{x\sqrt{a^2-x^2}}{a^2+x^2}\,dx$

32. $\int \frac{1}{(x+2)\sqrt{x-1}}\,dx$

33. $\int \left[\log\log x+\frac{1}{(\log x)^2}\right]dx$

34. $\int \frac{1}{x^2-2x\cos\theta+1}\,dx$

35. $\int \frac{3+4\sin x+2\cos x}{3+2\sin x+\cos x}\,dx$

36. $\int \frac{1}{1+3e^x+2e^{2x}}\,dx$

37. $\int \frac{e^{2x}}{(e^x+1)^{1/4}}\,dx$

38. $\int \frac{1}{x^2(x^4+1)^{3/4}}\,dx$

39. $\int \sec^2 x\,\mathrm{cosec}^2 x\,dx$

40. $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

41. $\int \sqrt{\frac{a^2-x^2}{x^8}}\,dx$

42. $\int \frac{1}{x+\sqrt{x+1}}\,dx$

43. $\int \frac{1}{(a+b\cos x)^2}\,dx,\ a>b$

44. $\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}\,dx$

45. $\int \frac{x^4-2x^3+3x^2-x+3}{x^3-2x^2+3x}\,dx$

46. $\int \frac{x}{\sec x+1}\,dx$

47. $\int \frac{\sin x}{\sin(x-a)}\,dx$

48. $\int \frac{1}{\sin x+\sec x}-dx$

49. $\int \frac{1}{\sqrt{\sin^3 x\sin(x+a)}}-dx$

50. $\int \frac{\sqrt{\cos 2x}}{\sin x}\,dx$

51. $\int \frac{e^x(x^3-x+2)}{(1+x^2)^2}\,dx$

52. $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}\,dx$

53. $\int x\sin^{-1}\left[\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right]dx$

54. $\int \frac{x-1}{\left[(x+1)^3(x+2)^5\right]^{1/4}}\,dx$

55. $\int \frac{x^2}{(x\sin x+\cos x)^2}\,dx$

56. $\int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx$

57. $\int \frac{\sqrt{x^2+1}\,(\log(x^2+1)-2\log x)}{x^4}\,dx$

58. $\int \frac{\sin x}{\sin 4x}\,dx$

59. $\int \frac{(\cos x-\sin x)}{7-9\sin 2x}\,dx$

60. $\int \frac{1}{\cos(x+a)\cos(x+b)}\,dx$

NOW ATTEMPT OBJECTIVE WORKSHEET TO COMPLETE THIS EBOOK

## THINGS TO REMEMBER

**1.** Properties of Integration :

(i) $\int k\,f(x)dx = k\int f(x)\,dx$

(ii) $\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots\ldots \pm f_n(x)]dx$

$= \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \int f_3(x)dx \pm \ldots\ldots \pm \int f_n(x)dx$

(iii) $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

(iv) $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$

(v) $\int f(x)\,dx = g(x)$, then $\int f(ax+b)\,dx = \frac{1}{a}g(ax+b) + C$

**2.** Integrals of Basic Trigonometric Functions

(i) $\int \sin x\,dx = -\cos x + C$

(ii) $\int \cos x\,dx = \sin x + C$

(iii) $\int \sec^2 x\,dx = \tan x + C$

(iv) $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

(v) $\int \sec x\tan x\,dx = +\sec x + C$

(vi) $\int \operatorname{cosec} x\cot x\,dx = -\operatorname{cosec} x + C$

## 3. Integrals of Basic Inverse and Exponential Functions

(i) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C \,;\, |x| < 1$

(ii) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

(iii) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} |x| + C \,;\, |x| > 1$

(iv) $\int a^x dx = \frac{a^x}{\log a} + C \,;\, a \neq 1, a > 0$

(v) $\int e^x\, dx = e^x + C$

## 4. Further Generalisation of Some Results using property 1.2 (v)

(i) $\int (ax+b)^n\, dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C$

(ii) $\int \frac{dx}{(ax+b)^2} = \frac{1}{a}\left(\frac{-1}{ax+b}\right) + C$

(iii) $\int \frac{dx}{\sqrt{ax+b}} = \frac{1}{a} 2\sqrt{ax+b} + C$

(iv) $\int \frac{dx}{(ax+b)^p} = \frac{1}{a} \frac{-1}{(p-1)(ax+b)^{p-1}}$

(v) $\int \frac{1}{ax+b}\, dx = \frac{1}{a} \log|ax+b| + C$

(vi) $\int \sin(ax+b)\, dx = -\frac{1}{a} \cos(ax+b) + C$

(vii) $\int \cos(ax+b)\, dx = +\frac{1}{a} \sin(ax+b) + C$

(viii) $\int \sec^2(ax+b)\, dx = \frac{1}{a} \tan(ax+b) + C$

(ix) $\int \text{cosec}^2(ax+b)\, dx = -\frac{1}{a} \cot(ax+b) + C$

(x) $\int \sec(ax+b) \tan(ax+b)\, dx = \frac{1}{a} \sec(ax+b) + C$

(xi) $\int \text{cosec}(ax+b) \cot(ax+b)\, dx = -\frac{1}{a} \text{cosec}(ax+b) + C$

(xii) $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$

## 5. Integrals of tan *x*, cot *x*, sec *x*, cosec *x*

$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$ [Also given in Section 1.2 (iii)]

i.e. $\int \frac{\text{derivative of denominator}}{\text{denominator}} dx = \log|\text{denominator}| + C$

**6.** If $g(x)$ is a continuous and differentiable function, then to evaluate the integral of the form

$\int f[g(x)]\, g'(x)\, dx$, we substitute $g(x) = t$ and $g'(x)\, dx = dt$.

The substitution reduces the integral $\int f[g(x)]\, g'(x)\, dx$ to the form $\int f(t)\, dt$ i.e.

**7.** When the integrand is of the form $\tan^n x \sec^n x$, the following cases arise :

(a) When $m$ is even positive integer, substitute $\tan x = z$.

(b) When $n$ is odd positive integer, substitute $\sec x = z$.

**8.** When the integrand is of the form cotnx cosecmx, the folliwng cases arise :

(a) When $m$ is even positive integer, substitute $\cot x = z$.

(b) When $n$ is odd positive integer, substitute cosec $= z$.

**9.** When the integrand is of the form $\sin^m x \cos^n x$, the following cases arise :

(a) When $m$ is odd positive integer, substitute $\cos x = z$.

(b) When $n$ is odd positive integer, substitute $\sin x = z$.

## 10. Important Substitutions

| Expression | Substitution |
|---|---|
| $a^2 + x^2$ | $x = a\tan\theta$ or $x = a\cot\theta$ |
| $a^2 - x^2$ | $x = a\sin\theta$ or $x = a\cos\theta$ |
| $x^2 - a^2$ | $x = a\sec\theta$ or $x = a\,\text{cosec}\theta$ |
| $\sqrt{\dfrac{a-x}{a+x}}$ or $\sqrt{\dfrac{a+x}{a-x}}$ | $x = a\cos 2\theta$ |
| $\sqrt{\dfrac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$ | $x = \alpha\cos^2\theta + \beta\sin^2\theta$ |

## 11. Some standard results derived using substitution method

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$ [use $x = \tan\theta$ or $x = \cot\theta$ to derive the result]

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x+a}{x-a}\right| + C$$ [use $x = a\sec\theta$ or $x = \text{cosec}\theta$ to derive the result]

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ell og\left|\frac{a+x}{a-x}\right| + C$$ [use $x = a\cos\theta$ or $x = a\sin\theta$ to derive the result]

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$$ [use $x = a\tan\theta$ or $x = \cot\theta$ to derive the result]

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$$ [use $x = a\sec\theta$ or $x = a\,\text{cosec}\theta$ to derive the result]

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$ [use = $x$ a sin θ or $x$ = a cos θ to derive the result]

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$ [use $x = a\ sec\theta$ or $x = a\ cosec\ \theta$ to derive the result]

$$\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2 + x^2}\right| + C$$ [Use $x = tan\ \theta$ or $x = cot\ \theta$ to drive the result]

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$ [Use $x = sec\ \theta$ or $x = cosec\ \theta$ to derive the result]

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$ [Use $x = sin\ \theta$ or $x = \ cos\ \theta$ to derive the result]

**12.** Integrals of the type $\int \frac{1}{ax^2 + bx + c}dx, \int \frac{1}{\sqrt{ax^2\ bx + c}}dx, \int \sqrt{ax^2 + bx + c}\ dx$ .

To evaluate this type of integrals we express quadratic expression $ax^2 + bx + c$ in the form of perfect square and then apply the formulas given in section 2.3.

**13.** Integrals of the type :

(a) $\int \frac{px + q}{ax^2 + bx + c}dx,$ (b) $\int \frac{px + q}{\sqrt{ax^2 + bx + c}}dx,$ (c) $\int \frac{p(x)}{ax^2\ bx + c}dx$

where $p(x)$ is a polynomial of degree greater than or equal to 2.

To evaluate the integrals of type $(a)$ and $(b)$ we express the linear factor in the numerator as :

$px + q = m$ (derivative of quadratic expression in the denominator) $+ n$

(where $m$ and $n$ are unknown constant to be determined by equating the coefficients of $x$ and constant terms on both sides)

**14.** Integrals of the type

$$\int \frac{dx}{a\sin^2 x + b\cos^2 x}, \int \frac{dx}{a + b\sin^2 x}, \int \frac{dx}{a + b\cos x},$$

$$\int \frac{dx}{(a\sin x + b\cos x)^2}, \int \frac{f(\tan x)\, dx}{a\, b\sin^2 x + \cos^2 x + d\sin x\cos x}$$ where $f(\tan x)$ is a polynomial in $\tan x$.

To evaluate this type of integrals we proceed in the following manner :

STEP - I Divide numerator and denominator both by $\cos^2 x$.

STEP - II Put $\tan x = t$, $\sec^2 x\, dx = dt$.

This substiution will convert the trigonometric integral into algebric integral.

After employing these steps the integral will reduce to the form $\int \frac{f(t)dt}{At^2 + B + C}$, where $f(t)$ is a polynomial in $t$.

This integral can be evaluated by methods discussed in section 2.41 and 2.42.

**15.** Integrals of the type $\int \frac{dx}{a\sin x + b\cos x}$, $\int \frac{dx}{a + b\sin x}$, $\int \frac{dx}{a + b\cos x}$, $\int \frac{f\left(\tan\frac{x}{2}\right)dx}{a\sin x + b\cos x + c}$

where $f(\tan x/2)$ is a polynomial in $\tan x/2$.

To evaluate this type of integrals we proceed in the following manner :

STEP - I Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

STEP - II Replace $1 + \tan^2 x/2$ in the numerator by $\sec^2 x/2$.

STEP - III Put $\tan x/2 = t$ and $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$

After performing these three steps the integral reduces to the form $\int \frac{f(t)\,dt}{At^2 + BT + C}$ where $f(t)$ is a polynomial in $t$. This integral can be evaluated by methods discussed in section 2.41 and 2.42.

**16.** Integrals of the type $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx$.

To evaluate this type of integrals express numerator as follows :

Numerator = $m$ (Derivative of Denominator) + $n$ (denominator)

$\Rightarrow \quad (a\sin x + b\cos x) = m(c\cos x - d\sin x) + n(c\sin x + d\cos x)$

where $m$ and $n$ are constants to be determined by comparing the coefficients of $\sin x$ and $\cos x$ on both sides.

Therefore,

$$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx = \int \frac{m(c\cos x - d\sin x) + n(c\sin x + d\cos x)}{c\sin x + d\cos x}dx$$

$$= m\int \frac{\frac{d}{dx}(c\sin x + d\cos x)}{c\sin x + d\cos x}dx + n\int dx$$

**17.** Integrals of the type $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx$.

To evaluate this type of integrals express numerator as follows :

Numerator = $l$ (Derivative of Denominator) + $m$ (denominator) + $n$

$\Rightarrow$ $a$ sin$x$ + $b$ cos$x$ + $c$ = $l$ ($c$ cos$x$ – $q$ sin$x$) + $m$ ($p$ sin$x$ + $q$ cos$x$ + $r$) + $n$

where $l$, $m$ and $n$ are constants to be determined by comparing the coefficients of sin$x$, cos$x$ and constant terms on both sides.

Therefore,

$$\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx = \int \frac{l\,(p\sin x - q\cos x + r) + m\,(p\sin x + q\cos + r) + n}{p\sin x + q\cos x + r}\,dx$$

$$= l\int \frac{\frac{d}{dx}(p\sin x + q\cos x + r)}{p\sin x + q\cos x + r}\,dx + m\int dx + n\int \frac{dx}{p\sin x + q\cos x + r}$$

$$= l\,ln\,|\,p\sin x + q\cos x + r\,| + mx + n\int \frac{dx}{p\sin x + q\cos x + r} + C$$

The last integral on RHS can be evaluated by the methods given in Section. 2.44.

**18.** If we take $u$ as part 1 and $dv$ as part 2, then the above result can be written as :

$$\int (\text{part 1})\,(\text{part 2}) = (\text{part 1})\,[\text{integral of part 2}] - \int [\text{integral of part 2}][\text{differential of part 1}]$$

**19.** Integrals of the Type $\int e^x\left[f(x) + f'(x)\right]dx$

Let $I = \int e^x\left[f(x) + f'(x)\right]dx$ $\Rightarrow$ $I = \int e^x\, f(x)\,dx + \int e^x f'(x)\,dx$

**20.** Integrals of the type : $\int \frac{dx}{\text{linear}\sqrt{\text{linear}}}$

To evaluate these type of integrals we substitute $\sqrt{\text{linear}} = t$. This subsitution will reduce the integral to one of the known forms.

**21.** Integral of the type : $\int \frac{dx}{\text{quadratic}\sqrt{\text{linear}}}$

To evaluate these type of integrals we subsitute $\sqrt{\text{linear}} = t$. This substitution will reduce the integral to one of the known forms.

**22.** Integral of the type : $\int \frac{dx}{(ax^2 + b)\sqrt{cx^2 + d}}$

To evaluate these type of integrals we substitute $x = \frac{1}{t}$. This substitution will reduce the intergral to one of the known forms.

**23.** Integral of the type : $\int \frac{dx}{\text{linear}\sqrt{\text{quadratic}}}$

To evluate these type of integrals we subtitute Linear $= \frac{1}{t}$. The substitution will reduce the integral to one of the know forms.

**24.** Integral of the type : $\int f\,(x, (ax+b)^{m_1/n_1}, (ax+b)^{m_2/n_2}, \ldots\ldots)\,dx$

where $f$ is a irrational function and $m_1$, $n_1$, $m_2$, $n_2$ are integers.
To evaluate this type of integral we transform it into an integral of rational function by substituting $ax + b = t^s$, where $s$ is the least common multiple of the numbers $n_1$, $n_2$, . . . . . .

**25.** Integral of the type : $\int x^m\,(a+bx^n)^p\,dx$,

where $m$, $n$ and $p$ are rational numbers.
To evaluate this type of integral observe the following steps :

(i) If $p$ is an integer, the integral reduces to the integral of a rational function by means of the substitution $x = t^s$, where $s$ is least common multiple of the denominator of the fractions $m$ and $n$.

(ii) If $(m + 1)/n$ is an integer, the integral can be rationlised by the substitution $a + bx^n = t^s$, where $s$ is the denominator of the fraction $p$.

(iii) If $(m + 1)/n + p$ is an integer, substitute $ax^{-n} + b = t^s$, where $s$ is the denominator of the fraction $p$.

**26.** Integration of irrational algebraic functions by trigonometric substitutions

Those Irrational functions which in involve either of $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ . . . . . . . terms and no other radical, can be integrated by a suitable trigonometric substitution. The trigonometric substitutions are similar to those given in Section 2.2.

## SOLUTIONS TO IN-CHAPTER EXERCISE - A

**(i)** $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx=\int\left(x+\frac{1}{x}-2\right)dx=\int xdx+\int\frac{1}{x}dx-\int 2dx=\frac{x^2}{2}+\ell nx-2x+C$

**(ii)** $\int\frac{x^3+3x^2+4}{\sqrt{x}}dx=\int x^{5/2}dx+\int 3x^{3/2}dx+\int 4x^{-1/2}dx=\frac{x^{7/2}}{7/2}+3.\frac{x^{5/2}}{5/2}+\frac{4x^{1/2}}{1/2}+C$

**(iii)** $\int\frac{dx}{\sqrt{ax+b}-\sqrt{ax+c}}=\int\frac{\sqrt{ax+b}+\sqrt{ax+c}}{b-c}dx=\frac{1}{(b-c)}\left(\frac{2}{3}\frac{(ax+b)^{3/2}}{a}+\frac{2}{3}\frac{(ax+c)^{3/2}}{a}\right)+C$

**(iv)** $\int\frac{x^3+3x^2+2x+1}{x-1}dx=\int\left(x^2+4x+6\right)dx+\int\frac{7}{x-1}dx=\frac{x^3}{3}+\frac{4x^2}{2}+6x+7\log|x-1|+C$

**(v)** $\int\cos^3 xdx=\int\frac{\cos 3x+3\cos x}{4}dx=\frac{1}{4}\left[\frac{1}{3}\sin 3x+3\sin x\right]+c$

**(vi)** $\int(\tan x+\cot x)^2 dx=\int\left(\tan^2 x+\cot^2 x+2\right)dx=\int\left(\sec^2 x+\operatorname{cosec}^2 x\right)dx=\tan x-\cot x+C$

**(vii)** $\int\sin x\sqrt{1-\cos 2x}dx=\int\sin x.\sqrt{2}(\sin x)dx=\sqrt{2}\int\sin^2 xdx=\sqrt{2}\times\frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]+C$

**(viii)** $\int\left(e^{a\log x}+e^{x\log a}\right)dx=\int\left(x^a+a^x\right)dx=\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+C$

**(ix)** $\int\cos x.\cos 2x.\cos 3x\,dx=\int\frac{1}{2}(\cos 3x+\cos x)\cos 3xdx=\frac{1}{2}\left[\int\cos^2 3xdx+\int\cos x\cos 3xdx\right]$

$=\frac{1}{4}\left[\int(1+\cos 6x)dx+\int\cos 4xdx+\int\cos 2xdx\right]=\frac{1}{4}\left(x+\frac{\sin 6x}{6}+\frac{\sin 4x}{4}+\frac{\sin 2x}{2}\right)+C$

**(x)** $\int(1+x)\sqrt{1-x}\,dx=-\int(-1-x)\sqrt{1-x}\,dx=-\int(-2+1-x)\sqrt{1-x}\,dx=\int 2\sqrt{1-x}\,dx-\int(1-x)^{3/2}dx$

$=-\frac{4}{3}(1-x)^{3/2}+\frac{2}{5}(1-x)^{5/2}+C$

**(xi)** $\int\frac{2x-3}{\sqrt{2x+1}}dx=\int\sqrt{2x+1}-4\int(2x+1)^{-1/2}dx=\frac{(2x+1)^{3/2}}{\frac{3}{2}.2}-4\frac{(2x+1)^{1/2}}{\frac{1}{2}.2}+C$

**(xii)** $\int\frac{\sin x+\cos x}{\sqrt{1+\sin 2x}}dx=\int\frac{\sin x+\cos x}{\sqrt{(\sin x+\cos x)^2}}dx=\int\frac{\sin x+\cos x}{|\sin x+\cos x|}dx$

$$= \begin{cases} x + C & \forall\ x \in \left(2n\pi - \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4}\right) \\ -x + C & \forall\ x \in \left(2n\pi + \frac{3\pi}{4}, 2n\pi + \frac{7\pi}{4}\right) \end{cases}$$

**(xiii)** $\int \sin x \cos x\,(\sin 2x + \cos 2x)\,dx = \frac{1}{2}\int \sin^2 2x\,dx + \frac{1}{4}\int \sin 4x\,dx = \frac{1}{4}\left(x - \frac{\sin 4x}{4} - \frac{\cos 4x}{4}\right) + C$

**(xiv)** $\int \frac{\sin^2 x}{(1+\cos x)^2}dx = \int \frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{4\cos^4\frac{x}{2}}dx = \int \tan^2\frac{x}{2}dx = \int\left(\sec^2\frac{x}{2} - 1\right)dx = \frac{\tan\frac{x}{2}}{1/2} - x + C$

**(xv)** $\int \frac{1}{1-\sin x}dx = \int \frac{1+\sin x}{\cos^2 x}dx = \int \sec^2 x\,dx + \int \sec x \tan x\,dx = \tan x + \sec x + C$

**(xvi)** $\int \frac{1}{1-\cos x}dx = \int \frac{1+\cos x}{\sin^2 x}dx = \int \operatorname{cosec}^2 x\,dx + \int \operatorname{cosec} x.\cot x\ dx + C = -\cot x - \operatorname{cosec} x + C$

## SOLUTIONS TO IN-CHAPTER EXERCISE - B

**(i)** $\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x}+1}$ ; Let $e^x = t \Rightarrow e^x\,dx = dt$

$= \int \frac{dt}{t^2+1} = \tan^{-1} t + c = \tan^{-1} e^x + C$

**(ii)** Let $1 - x^3 = t \Rightarrow -3x^2\,dx = dt$

$\int \frac{x^5 dx}{\left(1-x^3\right)^2} = \int \frac{(1-t).dt}{t^2}\left(-\frac{1}{3}\right) = \int \frac{(t-1)}{3t^2}\,dt = \frac{1}{3}\,\ell n\,|t| + \frac{1}{3t} + C$

$= \frac{1}{3}\,\ell n\,|1 - x^3| + \frac{1}{3\left(1 - x^3\right)} + C$

**(iii)** $\int \frac{x^3}{\sqrt{x^8+1}}dx$ Let $x^4 = t \Rightarrow 4x^3\,dx = dt$

$= \frac{1}{4}\int \frac{dt}{\sqrt{1+t^2}} = \frac{1}{4}\,\ell n\left|x^4 + \sqrt{1+x^8}\right| + C$

**(iv)** $\int \frac{2x\tan^{-1}x^2}{1+x^4}dx = \int t\,dt = \frac{t^2}{2}+C$ $\left[\text{Put } \tan^{-1}x^2 = t \Rightarrow \frac{2x}{1+x^4}dx = dt\right]$

$= \frac{\left(\tan^{-1}x^2\right)^2}{2}+C$

**(v)** $\int \frac{\sin x+\cos x}{(\sin x-\cos x)^3}dx = \int \frac{dt}{t^3} = \frac{-1}{2t^2}+C$ [Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\,dx = dt$]

$= \frac{-1}{2(\sin x-\cos x)^2}+C$

**(vi)** $\int \frac{xdx}{\left(ax^2+b\right)^p} = \frac{1}{2a}\int \frac{dt}{t^p} = \frac{-1}{2a(p-1)t^{p-1}}+c = \frac{1}{2a}(1-p)^{-1}\left(ax^2+b\right)^{1-p} + C$

[Put $ax^2 + b = t \Rightarrow 2ax\,dx = dt$]

**(vii)** $\int \frac{\cos^3 x}{\sqrt{\sin x}}dx$ Let $\sin x = t^2 \Rightarrow \cos x\,dx = 2t\,dt$

$= \int \frac{\left(1-t^4\right)}{t}\,.\,2tdt = 2t-\frac{2t^5}{5}+c = 2\sqrt{\sin x}-\frac{2}{5}(\sin x)^{5/2}+c$

**(viii)** $\int \cos x\,.\,\tan^3 x\,dx$ Let $\cos x = t \Rightarrow -\sin x dx = dt$

$= \int \sin x\,.\frac{\sin^2 x}{\cos^2 x}dx = -\int \frac{\left(1-t^2\right)}{t^2}dt = \frac{1}{t}+t+c = \sec x+\cos x+c$

**(ix)** $\int \tan x\,.\sec^6 x\,dx = \int t^5 dt = \frac{t^6}{6}+c = \frac{\sec^6 x}{6}+c$ [Let $\sec x = t \Rightarrow \sec x.\tan x dx = dt$]

**(x)** $\int \tan^3 x.\sec^5 x\,dx = \int \left(t^2-1\right).t^4\,dt$ [Let $\sec x = t \Rightarrow \sec x.\tan x dx = dt$]

$= \int \left(t^6-t^4\right)dt = \frac{t^7}{7}-\frac{t^5}{5}+c = \frac{\sec^7 x}{7}-\frac{\sec^5 x}{5}+c$

**(xi)** $\int \frac{dx}{x\cos^2(\log x)} = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\,dt$ [Let $\log x = t \Rightarrow \frac{1}{x}\,.\,dx = dt$]

$= \tan t + c = \tan(\log x) + c$

**(xii)** $\int \frac{x^3dx}{\left(x^2-a^2\right)^{3/2}} = \int \frac{\left(t^2+a^2\right).tdt}{t^3}$ [Let $x^2 - a^2 = t^2 \Rightarrow 2xdx = 2tdt$]

$= \int \left(1+\frac{a^2}{t^2}\right)dt = t-\frac{a^2}{t}+c = \sqrt{x^2-a^2}-\frac{a^2}{\sqrt{x^2-a^2}}+c$

## SOLUTIONS TO IN-CHAPTER EXERCISE - C

**(i), (ii), (iii)** - Similar to illustration - 17 ; **(iv)** Similar to illustration - 18

**(v)** Similar to illustration - 19 ; **(vi)** Similar to illustration - 20 (ii)

**(vii)** Similar to illustration - 21

**(viii)** $2 + 3\cos x = \ell(\sin x + 2\cos x + 3) + m(\cos x - 2\sin x) + n$

Comparing coefficients of $\sin x$, $\cos x$ and const. term,we get :

$2 = 3\ell + n\ ;\ 3 = 2\ell + m\ ;\ \ell - 2m = 0 \quad \Rightarrow \quad m = \frac{3}{5}\ ;\ \ell = \frac{6}{5}\ ;\ n = -\frac{8}{5}$

$$\Rightarrow \quad \int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}\,dx = \frac{6}{5}\int dx + \frac{3}{5}\int \frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3}\,dx - \frac{8}{5}\int \frac{dx}{\sin x + 2\cos x + 3}$$

$$= \frac{6}{5}x + \frac{3}{5}\log|\sin x + 2\cos x + 3| - \frac{8}{5}\int \frac{dx}{\sin x + 2\cos x + 3}$$

Let $I_1 = \int \frac{dx}{\sin x + 2\cos x + 3}$

Put $\sin x = \frac{2\tan x/2}{1 + \tan^2 x/2}$ ; $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$ and Let $\tan x/2 = t$

$$\Rightarrow \quad I_1 = \int \frac{dx}{\sin x + 2\cos x + 3} = \int \frac{\sec^2 x/2\,dx}{2\tan\frac{x}{2} + 2 - 2\tan^2\frac{x}{2} + 3 + 3\tan^2 x/2} = \int \frac{\sec^2 x/2\ dx}{\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 5}$$

Substitute $\tan\frac{x}{2} = t \ \Rightarrow\ \sec^2\frac{x}{2}dx = 2dt \ \Rightarrow\ I_1 = \tan^{-1}\left(\frac{1 + \tan x/2}{2}\right) + c$

**(ix)** $\int \frac{dx}{(2\sin x + 3\cos x)^2} = \int \frac{dx}{\left(4\sin^2 x + 9\cos^2 x + 12\sin x\cos x\right)}$ [Proceed similar to (v)]

**(x)** $\int \frac{dx}{\cos x(\sin x + 2\cos x)} = \int \frac{dx}{\cos x\sin x + 2\cos^2 x}$ [Proceed similar to (v)]

**(xi)** $\int \frac{x^2dx}{x^4 + 1} = \frac{1}{2}\int \frac{2x^2dx}{1 + x^4} = \frac{1}{2}\int \frac{x^2 + 1}{1 + x^4}dx + \frac{1}{2}\int \frac{x^2 - 1}{1 + x^4}dx$ [Proceed as in Illustration - 22]

**(xii)** $\int \frac{dx}{x^4 + 1} = \frac{1}{2}\int \frac{2dx}{1 + x^4} = \frac{1}{2}\int \frac{1 + x^2}{1 + x^4}dx + \frac{1}{2}\int \frac{1 - x^2}{1 + x^4}dx$ [Proceed as in Illustration - 22]

**(xiii)** $\int \sqrt{\tan x}\, dx = \int \frac{\sqrt{\tan x}.\sec^2 x}{1+\tan^2 x} dx = \int \frac{t.2t}{1+t^4} dt$ [Put $\tan x = t^2 \Rightarrow \sec^2 x . dx = 2t dt$]

$= \int \frac{t^2+1}{1+t^4} dt + \int \frac{t^2-1}{t^4+1} dt$ [Proceed as in Illustrations - 22]

**(xiv)** Substitute $\cot x = t^2$ and solve.

**(xv)** $\int \frac{dx}{\sin^4 x + \cos^4 x} = \int \frac{\sec^4 x dx}{1+\tan^4 x} = \int \frac{\left(1+\tan^2 x\right)\sec^2 x}{\left(1+\tan^4 x\right)} dx$ [Put $\tan x = t \Rightarrow \sec^2 x dx = dt$]

$= \int \frac{\left(1+t^2\right).dt}{1+t^4}$ [Proceed as in Illustrations - 22]

## SOLUTIONS TO IN-CHAPTER EXERCISE - D

**(i)** Apply By-parts taking $\log(1+x)$ as first part.

$$\int x \log(x+1) dx = \frac{x^2}{2}\log(x+1) - \frac{1}{2}\log(x+1) + \frac{x}{2} - \frac{x^2}{4} + c$$

**(ii)** Apply By-parts taking $\log x$ as first part. $\int \frac{\log x}{x^2} dx = -\frac{\log x}{x} - \frac{1}{x} + c$

**(iii)** $\int \frac{x}{1+\sin x} dx = \int \frac{x(1-\sin x)}{\cos^2 x} dx = \int x \sec^2 x dx - \int x \tan x \sec x dx$

Apply By - parts taking '$x$' as the first part in both the integrations.

$= x\tan x - \ell n|\sec x| - \left(x \sec x - \ell n|\sec x + \tan x|\right) + c$

**(iv)** Apply By - parts taking $\cos(\log x)$ as first part and $dx$ as second part.

$$\Rightarrow \quad \int \cos(\log x) dx = \frac{x}{2}\left(\cos(\log x) + \sin(\log x)\right)$$

**(v)** Apply By - parts taking $\log\left(x+\sqrt{x^2+a^2}\right)$ as first part and $dx$ as second part.

**(vi)** Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t . dt$

$$\Rightarrow \quad I = \int \frac{\log x}{(1+\log x)^2} dx = \int \frac{t.e^t}{(1+t)^2} dt = \int e^t \left[\frac{1}{1+t} - \frac{1}{(1+t)^2}\right] dt = \frac{e^t}{t+1} + c = \frac{x}{1+\log x} + c$$

**(vii)** Misprinting : $I = \int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx = \int e^x \left(\frac{1+x^2-2x}{\left(1+x^2\right)^2}\right) dx = \int e^x \left(\frac{1}{1+x^2} - \frac{2x}{\left(1+x^2\right)^2}\right) dx = \frac{e^x}{1+x^2} + c$

**(viii)** $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx=\int e^x\left(\frac{\operatorname{cosec}^2 x/2}{2}-\cot\frac{x}{2}\right)dx=-e^x\cot\frac{x}{2}+c$

$\left[\because \int e^x\left(f(x)+f'(x)\right)dx=e^x f(x)+c\right]$

**(ix)** $\int\sqrt{x}\,(\log x)^2\,dx=(\log x)^2\times\frac{2}{3}x^{3/2}-\int 2\log x\times\frac{1}{x}\times\frac{2}{3}x^{3/2}\,dx$

$=\frac{2}{3}x^{3/2}(\log x)^2-\frac{4}{3}\left(\log x\times\frac{2}{3}x^{3/2}-\int\frac{1}{x}\times\frac{2}{3}x^{3/2}dx\right)$

$=\frac{2}{3}x^{3/2}(\log x)^2-\frac{8}{9}x^{3/2}\log x+\frac{16}{27}x^{3/2}+c$

**(x)** Apply by - parts taking $x^2$ as first part.

**(xi)** Apply by - parts taking $\log\left(\sqrt{1-x}+\sqrt{1+x}\right)$ as first part and $dx$ as second part.

$\int\log\left(\sqrt{1-x}+\sqrt{1+x}\right)dx=x\log\left(\sqrt{1-x}+\sqrt{1+x}\right)+\frac{x}{2}\frac{-1}{2}\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}\frac{x}{2}\right]$

**(xii)** Apply by - parts taking $\log\left(\tan\frac{x}{2}\right)$ as first part.

$\int\cos x\log\left(\tan\frac{x}{2}\right)dx=\sin x\log\left(\tan\frac{x}{2}\right)-x+c$

## SOLUTIONS - IN-CHAPTER EXERCISE - E

**(i)** Similar to illustration - 33

**(ii)** $\int \frac{(3x+5)\,dx}{(x-1)^2(x+1)} \Rightarrow \frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$ Find $A$, $B$, $C$ and integrate.

**(iii)** $\frac{x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$, Find $A$, $B$, $C$, and integrate.

**(iv)** $\int \frac{(x-1)\,dx}{(x+1)(x^2+1)} = \int \frac{x(1+x)-(1+x^2)}{(x+1)(x^2+1)}dx = \int \frac{x}{x^2+1}dx - \int \frac{dx}{1+x} = \frac{1}{2}log|x^2+1| - log|1+x| + c$

**(v)** $\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$. Find $A$, $B$, $C$ and integrate

**(vi)** $\int \frac{x^4-1+1}{(x-1)(x^2+1)}dx = \int \frac{(x^2-1)(x^2+1)}{(x-1)(x^2+1)}dx + \int \frac{dx}{(x-1)(x^2+1)}$

$= \int (x+1)\,dx + \int \frac{dx}{(x-1)(x^2+1)}$

$= \frac{x^2}{2} + x + I_1$ where $I_1$ is same as (v)th part.

**(vii)** $\int \frac{x}{(x^2-a^2)(x^2-b^2)}dx = \frac{1}{a^2-b^2}\int x\left[\frac{1}{x^2-a^2} - \frac{1}{x^2-b^2}\right]dx = \frac{1}{2(a^2-b^2)} ln\left|\frac{x^2-a^2}{x^2-b^2}\right| + c$

**(viii)** Put $x^2 = t$ and then make partial fractions.

$\frac{(t+1)}{(t+2)(2t+1)} = \frac{A}{t+2} + \frac{B}{2t+1}$ ; $B = \left.\frac{t+1}{2t+1}\right|_{t=-2} = \frac{1}{3}$ ; $A = \left.\frac{t+1}{t+2}\right|_{t=-\frac{1}{2}} = \frac{1}{3}$

$\Rightarrow \frac{x^2+1}{(x^2+2)(2x^2+1)} = \frac{1}{3}\left[\frac{1}{x^2+2} + \frac{1}{2x^2+1}\right]$

Integrate both sides $\Rightarrow$ $I = \frac{1}{3}\cdot\frac{1}{\sqrt{2}}tan^{-1}\frac{x}{\sqrt{2}} + \frac{1}{6}\sqrt{2}\,tan^{-1}\sqrt{2}\,x + c$

**(ix)** $\int \frac{1+x+x^2}{x^3(1+x)}dx = \int \frac{1}{x^3} + \int \frac{dx}{x(x+1)} = -\frac{1}{2x^2} + \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = -\frac{1}{2x^2} + log|x| - log|x+1| + c$

## SOLUTIONS - IN-CHAPTER EXERCISE - F

**(i)** $\displaystyle\int \frac{x}{(x-3)\sqrt{x+1}}dx = \int \frac{(x-3)+3}{(x-3)\sqrt{x+1}}dx = \int \frac{dx}{\sqrt{x+1}} + 3\int \frac{dx}{(x-3)\sqrt{x+1}} = 2\sqrt{x+1}+3I_1$

To solve $I_1$, substitute $x+1=t^2$

$$\Rightarrow \quad 3\int \frac{dx}{(x-3)\sqrt{x+1}} = 3\int \frac{2dt}{t^2-4} = \frac{6}{4}\log\frac{t-2}{t+2}+c = \frac{3}{2}\log\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}+c$$

**(ii)** $\displaystyle I = \int \frac{2}{x\sqrt{1+x}}dx - 3\int \frac{x\,dx}{x\sqrt{1+x}} = 2\int \frac{2dt}{t^2-1} - 3\times 2\times\sqrt{1+x}+c$ [Substitute $x+1=t^2$]

**(iii)** $\displaystyle\int \frac{(x^2+2x+2)-2x}{(x^2+2x+2)\sqrt{x-1}} = \int \frac{dx}{\sqrt{x-1}} - 2\int \frac{xdx}{(x^2+2x+2)\sqrt{x-1}} = 2\sqrt{x-1}-2I_1$ ;

$$I_1 = \int \frac{xdx}{(x^2+2x+2)\sqrt{x-1}}$$

Let $x-1=t^2 \quad \Rightarrow \quad dx = 2tdt$

$$\int \frac{(t^2+1)(2tdt)}{\left[(t^2+1)^2+2(t^2+1)+2\right](t)} = 2\int \frac{(t^2+1)dt}{t^4+4t^2+5}$$

$$= \int \frac{\left(1+\frac{1}{\sqrt{5}}\right)(t^2+\sqrt{5})}{t^4+4t^2+5}dt + \int \frac{\left(1-\frac{1}{\sqrt{5}}\right)(t^2-\sqrt{5})}{t^4+4t^2+5}dt$$

$$= \left(1+\frac{1}{\sqrt{5}}\right)\int \frac{\left(1+\frac{\sqrt{5}}{t^2}\right)dt}{\left(t-\frac{\sqrt{5}}{t}\right)^2+(4+2\sqrt{5})} + \left(1-\frac{1}{\sqrt{5}}\right)\int \frac{\left(1-\frac{\sqrt{5}}{t^2}\right)dt}{\left(t+\frac{\sqrt{5}}{t}\right)^2-(2\sqrt{5}-4)}$$

substitute $t-\dfrac{\sqrt{5}}{t}=z \quad \Rightarrow \quad \left(1+\dfrac{\sqrt{5}}{t^2}\right)dt = dz$ ; substitute $t+\dfrac{\sqrt{5}}{t}=y \quad \Rightarrow \quad \left(1-\dfrac{\sqrt{5}}{t^2}\right)dt = dy$

**(iv)** $\displaystyle\int \frac{dx}{(x+1)^2\sqrt{\frac{x-1}{x+1}}}$ $\left[\text{Let } \dfrac{x-1}{x+1}=t^2\right] \Rightarrow \dfrac{(x+1)-(x-1)}{(x+1)^2}dx = 2tdt \Rightarrow \displaystyle\int \frac{tdt}{t} = \sqrt{\frac{x-1}{x+1}}+c$

**(v)** $\displaystyle I = \int \frac{x^2+2x+3}{(x+1)\sqrt{x^2+1}}dx = \int \frac{(x^2+2x+1)+2}{(x+1)\sqrt{x^2+1}}dx = \int \frac{dx}{\sqrt{x^2+1}} + \int \frac{xdx}{\sqrt{x^2+1}} + \int \frac{2dx}{(x+1)\sqrt{x^2+1}}$

$$\log\left|x+\sqrt{x^2+1}\right|+\sqrt{x^2+1}+\left(\int\frac{2dx}{(x+1)\sqrt{x^2+1}}=I_1\right)$$

To solve $I_1$, substitute $x+1=\dfrac{1}{t}$

**(vi)** Substitute $x+1=\dfrac{1}{t}$

**(vii)** Substitute $x^2+9=t^2$ $\Rightarrow$ $I=\dfrac{1}{2}\displaystyle\int\frac{2dt}{t^2-5}$

**(viii)** $\displaystyle\int\frac{\sqrt{1+x^2}}{x\,.\,x}\,xdx$ Let $1+x^2=t^2$ $\Rightarrow$ $xdx=tdt$

$$\int\frac{t\,(tdt)}{t^2-1}=\int\left(1+\frac{1}{t^2-1}\right)dt=t+\frac{1}{2}\,\ell n\left|\frac{t-1}{t+1}\right|+c$$

**(ix)** $$I=\int\frac{x^2}{\left(4+x^2\right)^{3/2}}dx=\int\frac{4+x^2-4}{\left(4+x^2\right)^{3/2}}dx=\int\frac{dx}{\sqrt{x^2+4}}-4\left(\int\frac{dx}{\left(x^2+4\right)\sqrt{x^2+4}}=I_1\right)$$

To solve $I_1$, substitute $x=\dfrac{1}{t}$

**(x)** Substitute $x^2=\cos 2\theta$

**(xi)** Substitute $x=t^6$ $\Rightarrow$ $I=\displaystyle\int\frac{\left(1+t^3\right)\left(6t^5\right)}{1+t^2}dt=6\int\frac{t^5+t^8}{1+t^2}dt$

On dividing Numerator and Denominator, we get : $I=\displaystyle\int\left(t^6-t^4+t^3+t^2-t-1+\frac{t+1}{t^2+1}\right)dt$

**(xii)** Substitute $x=t^2$ $\Rightarrow$ $dx=2tdt$

$\Rightarrow$ $I=\displaystyle\int\frac{2t^2.dt}{\left(1+t^2\right)\left(2+t^2\right)\left(3+t^2\right)}$ ; $\dfrac{2t^2}{\left(1+t^2\right)\left(2+t^2\right)\left(3+t^2\right)}=\dfrac{A}{1+t^2}+\dfrac{B}{2+t^2}+\dfrac{C}{3+t^2}$

Find $A$, $B$, $C$ and integrate.

## SOLUTIONS - IN-CHAPTER EXERCISE - G

**1.** **(i)** Similar to Illustration - 43. **(ii)** Similar to Illustration - 44

**2.** Similar to Illustration - 46 **3.** Similar to Illustration - 45

**4.** Put $tan\ x = t^2 \Rightarrow x = tan^{-1}\ t^2 \Rightarrow dx = \dfrac{2tdt}{1+t^4}$ $\Rightarrow$ $I = \int\left(t+\dfrac{1}{t}\right)\dfrac{2tdt}{1+t^4} = 2\int\dfrac{t^2+1}{t^4+1}dt$, then

use Illustration - 22(i)

**5.** Divide $N^r$ and $D^r$ by $cos^2 x$ and then put $tan\ x = t$

**6.** $\displaystyle\int\frac{sec^2\theta d\theta}{(tan\theta-2)(2tan\theta+1)}$ ($\because$ Divide $N$ and $D$ by $cos^2\theta$)

Let $tan\ \theta = t$ *and* then solve by partial fractions.

**7.** $\displaystyle\int\frac{dx}{p+qtanx} = \int\frac{cos\ x\ dx}{p\,cos\,x+q\,sin\,x}$. Express $cos\ x = l\ (p\ cos\ x + q\ sin\ x) + m\ (-p\ sin\ x + q\ cos\ x)$.

**8.** $\displaystyle I = \int\frac{dx}{sin\,x\,(sin\ x+2cos\ x)} = \int\frac{sec^2\ xdx}{tan\ x\ (tan\ x+2)} = \int\frac{dt}{t\ (t+2)}$. Use partial fractions.

**9.** $\displaystyle\int\frac{2dx}{2cos\ x+3} - \int\frac{sin\ x}{2\,cos\ x+3}\ dx = 2\int\frac{sec^2\frac{x}{2}}{2\left(1-tan^2\frac{x}{2}\right)+3\left(1+tan^2\frac{x}{2}\right)}dx + \frac{1}{2}\ \ln|\ 2\ cos\ x+3\ |+c$

Substitute $tan\dfrac{x}{2} = t$ in first integral and solve further.

**10.** $\displaystyle\int\frac{sin\,x}{(sin\,x-cos\,x)}\ dx = \frac{1}{2}\int\frac{(sin\,x-cos\,x)+(sin\,x+cos\,x)}{(sin\,x-cos\,x)}\ dx = \frac{x}{2}+\frac{1}{2}\ \ell n\ |\ sin\,x-cos\,x\ |+c$

**11.** $\displaystyle\int\frac{e^{2x}\ dx}{e^{2x}+1} = \frac{1}{2}\ \ln|\ e^{2x}+1|+c$

**12.** $\displaystyle I = \int\frac{a}{\sqrt{a^2-x^2}}\ dx + \int\frac{xdx}{\sqrt{a^2-x^2}} = a\ sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + c.$

**13.** $\displaystyle\int\frac{dx}{sin\ (x-a)\ sin\ (x-b)} = \frac{1}{sin\ (b-a)}\int\frac{sin\ (\overline{x-a}-\overline{x-b})}{sin\ (x-a)\ sin\ (x-b)}$

$\displaystyle = \frac{1}{sin\ (b-a)}\int\frac{sin(x-a)\ cos\ (x-b)-cos\ (x-a)\ sin\ (x-b)}{sin\ (x-a)\ sin\ (x-b)}$

$$= \frac{1}{\sin(b-a)} \int [\cot(x-b) - \cot(x-a)]\, dx = \frac{1}{\sin(b-a)} \ln \frac{\sin(x-b)}{\sin(x-a)} + c$$

**14.** **[ See INE - E (ii) ]**

**15.** Use partial fractions.

**16.** $\displaystyle\int \frac{dx}{\sin x\,(5+4\cos x)} = \int \frac{\sin x dx}{(1-\cos^2 x)(5+4\cos x)}$ ; Let $\cos x = t \Rightarrow I = -\displaystyle\int \frac{dt}{(1-t^2)(5+4t)}$.

Use partial fractions.

**17.** $\displaystyle\int x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\, dx = \int \frac{x\left(a^2-x^2\right)}{\sqrt{a^4-x^4}}\, dx$

Let $x^2 = t \quad \Rightarrow \quad 2xdx = dt$

$$\frac{1}{2}\int \frac{\left(a^2-t\right)}{\sqrt{a^4-t^2}}\, dt = \frac{a^2}{2}\int \frac{dt}{\sqrt{\left(a^2\right)^2-t^2}} - \frac{1}{2}\int \frac{tdt}{\sqrt{\left(a^2\right)^2-t^2}}$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{t}{a^2}\right) + \frac{1}{2}\sqrt{a^4-t^2} = \frac{a^2}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2}\sqrt{a^4-x^4} + c$$

**18.** Put $x = \sin\theta \Rightarrow I = \displaystyle\int \frac{\theta\cos\theta\, d\theta}{\cos^3\theta} = \int \theta \sec^2\theta\, d\theta$. Apply by parts.

**19.** $\displaystyle\int \left[1 + 2\tan^2 x + 2\tan x \sec x\right]^{1/2} dx = \int \left[(\sec x + \tan x)^2\right]^{1/2} dx = \int \sec x\, dx + \int \tan x\, dx$

**20.** Apply by parts taking $\log(1+x^2)$ as I part and $dx$ as second part.

**21.** Apply by-parts taking $\log(\sec x + \tan x)$ as I part and $\sin x$ as second part.

**22.** $\displaystyle\int \frac{\cos(x-a+a)}{\cos(x-a)}\, dx = \int \frac{\cos(x-a)\cos a - \sin(x-a)\sin a}{\cos(x-a)}\, dx = \int [\cos a - \tan(x-a)\sin a]\, dx$

**23.** $\displaystyle\frac{1}{\sin(b-a)}\int \frac{\sin(b-a)\, dx}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(b-a)}\int \frac{\sin(\overline{x-a} - \overline{x-b})}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(b-a)}$

$\displaystyle\int [\tan(x-a) - \tan(x-b)]\, dx$

**24.** Put $x = \cos\theta \Rightarrow I = -\displaystyle\int \frac{\theta}{2}\sin\theta\, d\theta$. Apply by parts.

**25.** Divide $N$ and $D$ by $\cos^4 x$. $I \displaystyle\int \frac{2\tan x \sec^2 x dx}{1+\tan^4 x} = \int \frac{dt}{1+t^2} = \tan^{-1} t + c$ [Substitute $t = \tan^2 x$]

**26.** $\displaystyle\int \frac{x-\sin x}{1-\cos x}\, dx = \int \frac{x dx}{2\sin^2\frac{x}{2}} - \int \frac{\sin x}{1-\cos x}\, dx = \frac{1}{2}\int x\, \mathrm{cosec}^2\frac{x}{2} dx - \ln|1-\cos x| + c.$

For the first integral, apply by parts and proceed further.

**27.** Let $\log x = t \Rightarrow \frac{1}{x} dx = dt \Rightarrow I = \int e^{-2t} t \, dt$. Apply by parts.

**28.** $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b} dx$ [Use rationalisation]

**29.** $\int \frac{1}{x \log x\left[(\log x)^2 - 3\log x - 10\right]} dx$. Put $\log x = t \Rightarrow I = \int \frac{dt}{t(t^2-3t-10)} = \int \frac{dt}{t(t-5)(t+2)}$

Use partial Fractions.

**30.** Rationalise to get : $I = \frac{1}{a-b}\left[\int x\sqrt{(x+a)}\,dx - \int x\sqrt{x+b}\,dx\right]$

Put $x+a=t^2$ Put $x+b=t^2$

**31.** Substitute $a^2 - x^2 = t^2 \Rightarrow I = -\int \frac{t^2 dt}{2a^2 - t^2} = \int\left(1 - \frac{2a^2}{2a^2-t^2}\right) dt = t + 2a^2 \int \frac{dt}{t^2 - \left(\sqrt{2}a\right)^2}$

**32.** Substitute $x - 1 = t^2$.

**33.** Let $\log x = t \Rightarrow \frac{1}{x} dx = dt \Rightarrow I = \int\left(\log t + \frac{1}{t^2}\right) e^t\,dt = \int\left(\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2}\right) e^t\,dt$

$\Rightarrow I = \int e^t\left(\log t + \frac{1}{t}\right) dt + \int e^t\left(-\frac{1}{t} + \frac{1}{t^2}\right) dt = e^t\left(\log t + \left(\frac{-1}{t}\right)\right) = x\left[\log\log x - \frac{1}{\log x}\right] + c.$

**34.** $\int \frac{dx}{(x-\cos\theta)^2 + \sin^2\theta} = \frac{1}{\sin\theta}\tan^{-1}\left(\frac{x-\cos\theta}{\sin\theta}\right) + c$

**35.** $\int \frac{2(3+2\sin x+\cos x)-3}{3+2\sin x+\cos x} dx = 2x - 3\int \frac{dx}{3+2\sin x+\cos x}$

$$= 2x - 3\int \frac{\sec^2\frac{x}{2}\,dx}{3+3\tan^2\frac{x}{2}+4\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}$$

$$= 2x - 3\int \frac{\frac{1}{2}\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}+2}$$ Let $\tan\frac{x}{2} = t \Rightarrow 2x - 3\int \frac{dt}{(t+1)^2+1}$

$$= 2x - 3\tan^{-1}\left(1+\tan\frac{x}{2}\right) + c$$

**36.** $\int \frac{e^x\,dx}{e^x(1+3e^x+2e^{2x})} = \int \frac{dt}{t(1+3t+2t^2)} = \int \frac{dt}{t(t+1)(2t+1)}$. Apply partial fractions.

**37.** Let $e^x + 1 = t^4 \Rightarrow e^x\,dx = 4\,t^3\,dt \Rightarrow I = \int \frac{(t^4-1)}{t}\,4\,t^3 dt$

**38.** $I = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$. Substitute $1 + \frac{1}{x^4} = t^4$.

**39.** $\int sec^2 x\, cosec^2 x dx = \int \frac{4}{sin^2 2x} dx = 4\int cosec^2 2x dx = 4\frac{(-cot\,2x)}{2} + c$

**40.** [Substitute $x = cos^2\theta$] $\Rightarrow$ $I = -2\int tan\frac{\theta}{2} cos\,\theta\, sin\,\theta d\theta = 2\int cos\,\theta\,(cos\,\theta - 1)\,d\theta$

**41.** $\int\sqrt{\frac{a^2}{x^2}-1}\,\frac{dx}{x^3}$. [Substitute $\frac{a^2}{x^2} - 1 = t^2$]

**42.** Substitute $x + 1 = t^2 \Rightarrow dx = 2t\,dt$ $\Rightarrow$ $I = \int\frac{2t\,dt}{t^2-1+t}$

**43.** $I = \int\frac{dx}{\left[a+b\left(\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}\right)\right]^2} = \int\frac{sec^4\frac{x}{2}dx}{\left[(a+b)+(a-b)\,tan^2\frac{x}{2}\right]^2}$. [Substitute $(a-b)\,tan^2\frac{x}{2}$ $= (a+b)\,tan^2\theta$ ]

$\Rightarrow$ $\sqrt{(a-b)}\,sec^2\frac{x}{2}dx = 2\sqrt{(a+b)}\,sec^2\theta\,d\theta$

$\Rightarrow$ $I = \int\frac{\left(1+tan^2\frac{x}{2}\right)sec^2\frac{x}{2}\,dx}{\left[a+b+(a-b)\,tan^2\frac{x}{2}\right]^2} = 2\int\frac{\left[1+\frac{a+b}{a-b}tan^2\theta\right]\sqrt{\frac{a+b}{a-b}}\,sec^2\theta\,d\theta}{(a+b)^2\,sec^4\theta}$

$\Rightarrow$ $I = \frac{2}{(a^2-b^2)^{3/2}}\int[(a-b)\,cos^2\theta + (a+b)\,sin^2\theta]\,d\theta$

$\Rightarrow$ $I = \frac{2}{(a^2-b^2)^{3/2}}\left[\frac{a-b}{2}\left(\theta+\frac{sin\,2\theta}{2}\right)+\frac{a+b}{2}\left(\theta-\frac{sin\,2\theta}{2}\right)\right]+c$

$\Rightarrow$ $I = \frac{2}{(a^2-b^2)^{3/2}}\left[a\theta - \frac{b}{2}sin\,2\theta\right] = \frac{2}{(a^2-b^2)^{3/2}}\left[a\theta - \frac{b\,tan\,\theta}{1+tan^2\theta}\right]+c$

Substitute $tan^2\theta = \frac{(a-b)}{(a+b)}\,tan^2\frac{x}{2}$ and simplify.

**44.** $I = \int \frac{x(x^2+1)+(x^2+2)}{(x^2+1)(x^2+2)}\,dx = \int \frac{x}{x^2+2}\,dx + \int \frac{dx}{x^2+1}$

**45.** $I = \int x\,dx + \int \frac{3-x}{x(x^2-2x+3)}\,dx$. Use partial fraction : $\frac{3-x}{x(x^2-2x+3)} = \frac{A}{x} + \frac{Bx+C}{x^2-2x+3}$

**46.** $\int \frac{x}{\sec x+1}\,dx = \int \frac{x}{\tan^2 x}(\sec x - 1)\,dx = \int x(\operatorname{cosec} x \cot x - \cot^2 x)\,dx$.

$= \int x \operatorname{cosec} x \cot x\,dx - \int x \operatorname{cosec}^2 x\,dx + \int x\,dx$ (Use by parts)

**47.** $\int \frac{\sin(x-a+a)}{\sin(x-a)}\,dx = \int \frac{\sin(x-a)\cos a + \cos(x-a)\sin a}{\sin(x-a)}\,dx = \cos a \int dx + \sin a \int \cot(x-a)\,dx$

**48.** We have $\sin 2x = (\sin x + \cos x)^2 - 1$ and $\sin 2x = 1 - (\sin x - \cos x)^2$

$$I = \int \frac{2\cos x\,dx}{\sin 2x + 2} = \int \frac{(\cos x + \sin x)\,dx}{\sin 2x + 2} + \int \frac{(\cos x - \sin x)\,dx}{\sin 2x + 2}$$

$$= \int \frac{(\cos x + \sin x)dx}{3 - (\sin x - \cos x)^2} + \int \frac{(\cos x - \sin x)\,dx}{(\sin x + \cos x)^2 + 1}$$

$$= \int \frac{dt}{3 - t^2} + \int \frac{dz}{z^2 + 1}$$

**49.** $\int \frac{\operatorname{cosec}^2 x\,dx}{\sqrt{\frac{\sin(x+a)}{\sin x}}} = \int \frac{\operatorname{cosec}^2 x\,dx}{\sqrt{\cos a + \cot x \sin a}} = \frac{-1}{\sin a}\int \frac{dt}{\sqrt{t}}$ where $t = \cos a + \cot x \sin a$

**50.** $\int \frac{\sqrt{\cos 2x} \times \sin x}{\sin x \times \sin x}\,dx = \int \left( \frac{\sqrt{2\cos^2 x - 1}}{1 - \cos^2 x} \sin x \right) dx$ Put $\cos x = t$

$$\int \frac{\sqrt{2t^2 - 1}\,(-dt)}{1 - t^2} = \int \frac{2t^2 - 1}{(t^2 - 1)\sqrt{2t^2 - 1}}\,dt$$

$$= 2\int \frac{1}{\sqrt{2t^2 - 1}}\,dt + \int \frac{dt}{(t^2 - 1)\sqrt{2t^2 - 1}}$$

**51.** Write $x^3 - x + 2 = (x^2 + 1)(x + 1) - x^2 - 2x + 1$

$$\Rightarrow I = \int e^x \left[ \underbrace{\frac{(x^2+1)(x+1)}{(x^2+1)^2}}_{f(x)} + \underbrace{\frac{1 - x^2 - 2x}{(x^2+1)^2}}_{f'(x)} \right] dx = \frac{e^x(x+1)}{x^2+1} + c$$

**52.** Let $x = t^2 \Rightarrow I = \int \frac{t^2 - 1 \quad 2t\,dt}{(t^2+1)\sqrt{t^6 + t^4 + t^2}} = \int \frac{(t^2 - 1)\quad 2dt}{(t^2+1)\sqrt{t^4 + t^2 + 1}}$

$$\Rightarrow I = 2\int \frac{\left(1-\frac{1}{t^2}\right)dt}{\left(t+\frac{1}{t}\right)\sqrt{t^2+1+\frac{1}{t^2}}}.\ \text{Now put } z = t+\frac{1}{t}$$

$$\Rightarrow I = 2\int \frac{dz}{z\sqrt{z^2-1}} = 2\sec^{-1}|z| = 2\sec^{-1}\left|\sqrt{x}+\frac{1}{\sqrt{x}}\right|.$$

**53.** Apply By parts taking $\sin^{-1}\left[\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right]$ as Ist part and $x$ as IInd part.

**54.** $$\int \frac{x-1\quad dx}{(x+1)^{3/4}\ (x+2)^{5/4}} = \int \frac{x-1}{\left[\frac{x+1}{x+2}\right]^{3/4}(x+2)^2}dx$$

$$\frac{x+1}{x+2} = t^4 \Rightarrow \frac{(x+2)-(x+1)}{(x+2)^2}dx = 4t^3dt \text{ and } x = \frac{2t^4-1}{1-t^4}$$

$$\int \frac{\left(\frac{3t^4-2}{1-t^4}\right)}{t^3}\left(4t^3dt\right) = \int \frac{3\left(t^4-1\right)+1}{1-t^4}dt = -3t - \frac{3}{2}\int\left[\frac{1}{t^2-1}-\frac{1}{t^2+1}\right]dt$$

**55.** $I = \int \frac{x}{\cos x}\frac{x\cos x}{(x\sin x+\cos x)^2}dx$. Use by parts.

$$= \frac{x}{\cos x}\int \frac{x\cos x\,dx}{(x\sin x+\cos x)^2} - \int \frac{-1}{(x\sin x+\cos x)}\left(\frac{\cos x + x\sin x}{\cos^2 x}\right)dx$$

$$= \frac{x}{\cos x}\left(\frac{-1}{(x\sin x+\cos x)}\right) + \tan x + C.\ \left[\text{Note that } \frac{d}{dx}(x\sin x+\cos x) = x\cos x\right]$$

**56.** $$I = \int \frac{\sin(x-\alpha)\,dx}{\sqrt{\sin(x+\alpha)\sin(x-a)}} = \underset{(\text{Substitute } t=\cos x)}{\int \frac{\sin x\cos\alpha\,dx}{\sqrt{\cos^2\alpha-\cos^2 x}}} - \underset{(\text{Substitute } z=\sin x)}{\int \frac{\cos x\sin\alpha}{\sqrt{\sin^2 x-\sin^2\alpha}}dx}$$

**57.** **[Similar to Q. 7 in OSE]**

**58.** $$I = \int \frac{dx}{4\cos x\cos 2x} = \int \frac{\cos x\,dx}{4(1-\sin^2 x)(1-2\sin^2 x)} = \int \frac{dt}{4(1-t^2)(1-2t^2)}.$$

**59.** Take $\cos x + \sin x = t$ and then solve.

**60.** **[Similar to Q. 23]**

# My Chapter Notes